

Essential Mathematics 1 – Vectors and vector operations

Vector quantities and vector operations may be applied in a variety of ways to represent, simplify and solve many problems in engineering science. Thus, vector representation and operations may be used to simplify and solve problems concerned with force systems, mechanical work, turning moments, displacement, velocity and acceleration, mechanisms, magnetic fields, uniform circular motion and gyroscopic motion, to name but a few. Here we present some of the underpinning geometry and algebraic operations, needed to use vectors to solve the sorts of engineering problems encountered in the areas, mentioned above.

1 Definition and notation

A *vector quantity* is one that has both magnitude and direction. For example, force, displacement, velocity, acceleration etc. Scalar quantities have only magnitude, for example, time, mass, energy etc.

Vector quantities are generally represented in standard ways, in this book, we will always represent vector quantities in bold faced type and their scalar magnitude by either two parallel lines enclosing the vector symbol or by use of standard italic type, the latter method being used where there is no chance of confusion with other variables. So for example we may represent the vector quantity as \mathbf{P} and its magnitude as $|\mathbf{P}|$ or as P .

2 Addition and subtraction of vectors

Addition of two vectors may be represented geometrically by the triangle rule, shown in Figure EM 1.1, where to add vector \mathbf{V}_2 to \mathbf{V}_1 , we place the tail of \mathbf{V}_2 at the head of \mathbf{V}_1 , the sum is then the vector \mathbf{V}_T from the tail of \mathbf{V}_1 to the head of \mathbf{V}_2 .

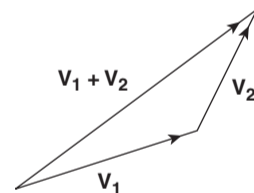


Figure EM 1.1 Vector addition

The difference $\mathbf{V}_1 - \mathbf{V}_2$ between the two vectors may be obtained by adding $-\mathbf{V}_2$ to \mathbf{V}_1 , note that the minus sign has the effect of reversing the direction of the vector, see Figure EM 1.2.

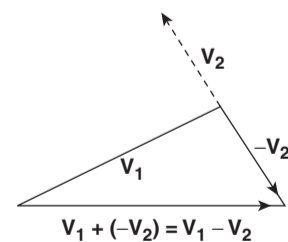


Figure EM 1.2 Vector subtraction

Figure EM 1.3 shows an equivalent way of determining the difference $\mathbf{V}_1 - \mathbf{V}_2$ by placing the *head* of \mathbf{V}_2 at the *head* of \mathbf{V}_1 .

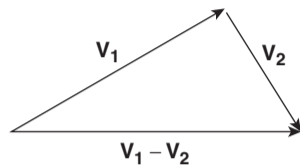


Figure EM 1.3 Alternative method for determining $\mathbf{V}_1 - \mathbf{V}_2$

It can now be seen from Figure EM 1.3 that $\mathbf{V}_1 - \mathbf{V}_2$ extends from the *tail* of \mathbf{V}_1 to the *tail* of \mathbf{V}_2 .

3 Dot or scalar product

The dot or scalar product $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$ as its name suggests, is used to find the scalar product of two vector quantities. For any two vectors \mathbf{A} and \mathbf{B} , this product may be found by taking the magnitude of \mathbf{A} multiplied by the component $B \cos \theta$ of \mathbf{B} (the projection of \mathbf{B}). This concept is illustrated in Figure EM 1.4.

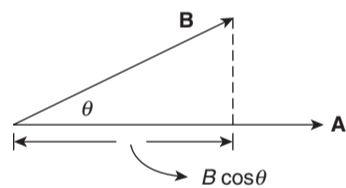


Figure EM 1.4 Illustration of $\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$

Example EM 1.1 Consider the packing case being hauled along by a force \mathbf{P} pulling on the rope, at an angle θ to the level horizontal floor, the case travelling over a magnitude \mathbf{d} (distance covered in a given direction).

Then, because both the force \mathbf{P} and the magnitude \mathbf{d} are vector quantities, the work done (WD) in hauling the packing case may be found using the dot product $\mathbf{P} \cdot \mathbf{d}$ or $WD = (P \cos \theta)d$, as shown in Figure EM 1.5.

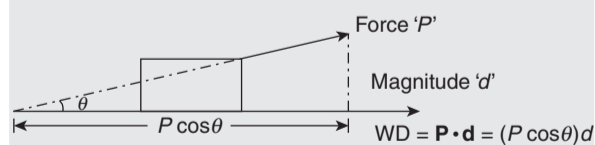


Figure EM 1.5 Work done, found by using the vector dot product

4 Unit vectors and vector components

One of the unusual consequences of the dot product of two non-zero vectors is that it equals zero or by virtue of the nature of the dot product, when $\mathbf{A} \cdot \mathbf{B} = 0$, these two vectors are mutually perpendicular, i.e. at right-angles or *orthogonal* to each other. We may use this fact when we consider the *direction of any vector* in three-dimensional space. To help us identify the position of any vector \mathbf{V} , we set up *unit vectors* (\mathbf{i} , \mathbf{j} , \mathbf{k}) on three-dimensional orthogonal Cartesian axes, where they indicate unit length along the x , y and z axes, respectively, Figure EM 1.6.

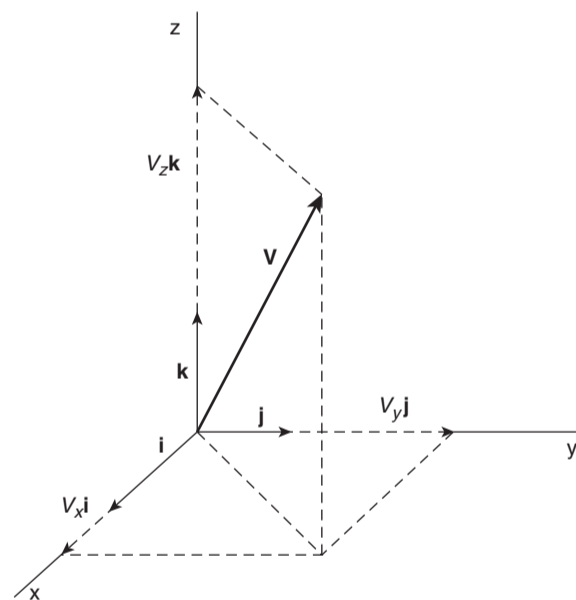


Figure EM 1.6 Three-dimensional unit vectors and vector components

We can identify the magnitude and direction of a vector \mathbf{V} , in three-dimensional space, by assigning values to the components of this vector (V_x , V_y and V_z) as shown in Figure EM 1.6. So that the *position of the vector* is given by $\mathbf{V} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k}$, look again at Figure EM 1.6, to see that this sum of the components along the x , y and z axes, result in the position of vector \mathbf{V} in space, given that its tail is positioned at $V_x \mathbf{i} = 0$, $V_y \mathbf{j} = 0$ and, $V_z \mathbf{k} = 0$. Note also that the sum of the vector components may be taken in any order.

Now you have already spent a great deal of time in two-dimensional space, finding for example, the vector force \mathbf{F} , given its two components F_x , and F_y , where to find the *magnitude* of this force you used Pythagoras theorem, i.e. $|\mathbf{F}|$ or $F = \sqrt{F_x^2 + F_y^2}$. This method for finding the *modulus* (*magnitude*) of a vector may be extended for three-dimensional space where the magnitude of the general vector $|\mathbf{V}| = V = \sqrt{V_x^2 + V_y^2 + V_z^2}$. Thus for example, the magnitude of a vector \mathbf{V} having a position in space given by its components (2, -4, 3), respectively along the x , y and z axes, would be

$$V = \sqrt{(2)^2 + (-4)^2 + (3)^2} = 5.385.$$

5 Direction cosines

In addition to fixing the position of an object in three-dimensional space by considering its *Cartesian components at its start and finish*, we may also find or fix its position by considering its *direction cosines*. Figure EM 1.7 shows the position vector system for a vector \mathbf{R} . Where the direction angles A , B and C are shown, it is the cosines of these angles that may be used to find the direction of vectors in three-dimensional space.

The method of finding these direction cosines is best illustrated by an example.

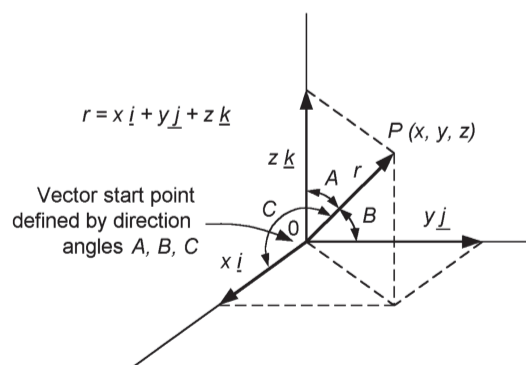


Figure EM 1.7 Position vector system

Example EM 1.2 Determine the magnitude and direction of the pole \mathbf{R} , shown in Figure EM 1.8.

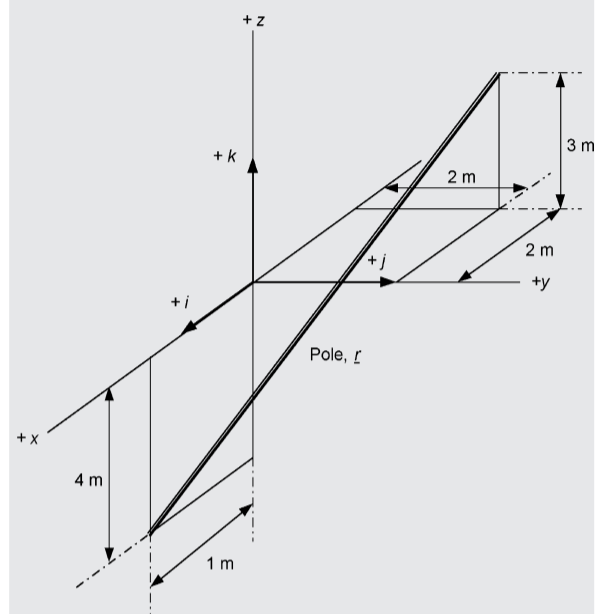


Figure EM 1.8 Location of pole \mathbf{R} in three-dimensional space

The vector representing the pole can be found by determining the vector co-ordinates of its start and finish points A and B, i.e.

$$A(1, 0, -4) \text{ and } B(-2, 2, 3).$$

Now $\mathbf{R} = \mathbf{R}_B - \mathbf{R}_A$ (in other words, position at finish minus position at start, defines the magnitude and position of \mathbf{R} . So using vector subtraction and the unit vectors in the x , y and z direction gives,

$$\mathbf{R} = (-2 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (3 - (-4))\mathbf{k},$$

so

$$\mathbf{R} = -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k},$$

then the magnitude of

$$|\mathbf{R}| = R = \sqrt{(-3)^2 + (2)^2 + (7)^2} = 7.9 \text{ m.}$$

Now the unit vector \mathbf{U} is defined as $\mathbf{U} = \frac{\mathbf{R}}{|\mathbf{R}|}$ that is the unit vector in the direction of \mathbf{R} is formed by dividing the individual components of \mathbf{R} by the magnitude of $|\mathbf{R}|$. This is identical to dividing a

vector by its scalar multiplier to obtain the unit vector, in this case the unit vector is,

$$\mathbf{U} = \frac{R}{|\mathbf{R}|} = \frac{-3}{7.9}\mathbf{i} + \frac{2}{7.9}\mathbf{j} + \frac{7}{7.9}\mathbf{k}.$$

Now the components of the unit vector give the cosines of the coordinate directions angles (A, B and C in Figure EM 1.7), of the pole from its start point. So these angles position the strut in its fixed direction as it leaves A for B. Then,

$$\cos \theta_i = -0.3797 \text{ so } \theta_i = 112.3^\circ,$$

$$\cos \theta_j = 0.2538 \text{ so } \theta_j = 75.3^\circ,$$

$$\cos \theta_k = 0.866 \text{ so } \theta_k = 30^\circ.$$

So we have determined both the magnitude and size of the pole and its direction in space, in terms of the direction cosines.

6 Vector algebra

Vector subtraction using algebra, rather than drawing has already been used in the previous example (EM 1.2), as well as the algebraic methods for finding the magnitude and the direction cosines. Here we summarise these algebraic rules, in terms of the equations for *vector components*, in three-dimensional space. Then for vectors \mathbf{A} and \mathbf{B} , we have:

$$\text{a) } \mathbf{A} + \mathbf{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$$

$$\text{b) } \mathbf{A} - \mathbf{B} = (A_x - B_x, A_y - B_y, A_z - B_z)$$

$$\text{c) } |\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\text{d) } \mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

Also from the definition of the dot product

$$\text{e) } \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \text{ since } \cos 0 = 1$$

$$\text{f) } \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0 \text{ since } \cos 90^\circ = 0, \text{ therefore since}$$

$$\mathbf{A} \cdot \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k})$$

Then,

$$\text{g) } \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \text{ Also}$$

$$\text{h) } \mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2$$

$$\text{i) } \text{Also from dot product } \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|}$$

Note that equation g) gives the dot product of two vectors in terms of the sum of their component products. While equation i) shows in a similar way, the method we used in Example EM 1.2 to find the *direction cosines* of the components of any vector \mathbf{A} , using the unit vector \mathbf{U} , which in general terms may be written as,

$$\mathbf{U} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_x}{|\mathbf{A}|}\mathbf{i} + \frac{A_y}{|\mathbf{A}|}\mathbf{j} + \frac{A_z}{|\mathbf{A}|}\mathbf{k}$$

where the quantities $\frac{A_x}{|\mathbf{A}|}$, $\frac{A_y}{|\mathbf{A}|}$ and $\frac{A_z}{|\mathbf{A}|}$ give the *cosine of the angles* in the x , y and z planes. The use of the above rules of vector algebra, are illustrated in the next example.

Example EM 1.3 Given that vector $\mathbf{A} = (2, -3, 4)$ and $\mathbf{B} = (3, 2, -5)$, determine:

$$\mathbf{A} + \mathbf{B}, \mathbf{A} - \mathbf{B}, |\mathbf{A}|, |\mathbf{B}|,$$

the dot product $\mathbf{A} \cdot \mathbf{B}$ and the angle between \mathbf{A} and \mathbf{B} .

Then

$$\mathbf{A} + \mathbf{B} = (2 + 3, -3 + 2, 4 - 5) = (5, -1, -1)$$

$$\mathbf{A} - \mathbf{B} = (2 - 3, -3 - 2, 4 - 5) = (-1, -5, 9)$$

$$|\mathbf{A}| = \sqrt{(2^2 + (-3)^2 + 4^2)} = \sqrt{29} = 5.385.$$

Do remember that any values under the modulus sign is positive, therefore it is not really correct to write -3 , it should just be 3 , even though all these values are squared and will therefore be positive no matter what the value of the integer.

$$|\mathbf{B}| = \sqrt{(3^2 + 2^2 + (-5)^2)} = 6.164$$

We will use formula (g),

$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$, to find the dot product then,

$$\mathbf{A} \cdot \mathbf{B} = (2)(3) + (-3)(2) + (4)(-5) = -20$$

To determine the angle between the two vectors **A** and **B**, we first find the cosine of this angle from the relationship,

$$\begin{aligned}\cos \theta &= \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} \text{ so that,} \\ \cos \theta &= \frac{-20}{(5.385)(6.164)} \\ &= -0.6025 \text{ and so the angle } \theta = 127^\circ.\end{aligned}$$

7 Cross product

The *cross* or *vector product* $\mathbf{A} \times \mathbf{B}$ results in the production of a third vector with a *magnitude* of $\mathbf{A} \times \mathbf{B} = AB \sin \theta$ and a *direction* specified by the right-hand screw rule as shown in Figure EM 1.9.

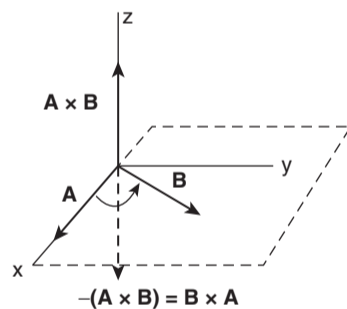


Figure EM 1.9 Direction of vector cross product given by right-hand screw rule

Note also, from Figure EM 1.9 that the minus cross product $-(\mathbf{A} \times \mathbf{B})$ reverses the direction of the vector formed and is equal to $(\mathbf{B} \times \mathbf{A})$.

In mechanics the vector cross product has a variety of applications including those related to torque or twisting moments. Consider a torque tube of radius **r** measured from the centre of rotation, acted upon by a resisting force **F** from say a valve, as shown in Figure EM 1.10.

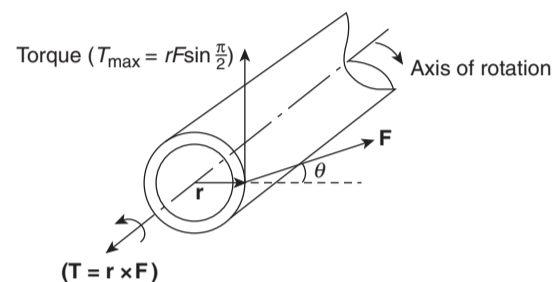


Figure EM 1.10 Vector analysis of a drive shaft

Now we may use the vector cross product to determine both the *magnitude* and *direction* of the torque resulting from the applied force, no matter where this force is applied on the radius of the cross-section.

Then by considering the radius **r** and force **F** as vectors, the magnitude and direction of the resulting torque may be found using the vector cross product $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ where the magnitude is found using, $|\mathbf{r} \times \mathbf{F}| = rF \sin \theta$. So as a rather trivial example of this particular application of the vector cross product let us assume that the torque shaft has an external diameter of 20 cm and the force, $F = 5 \text{ kN}$. Then the magnitude of the torque will occur at the extremity of the shaft, where **r** is a maximum and where $\theta = \pi/2$ radian. Then,

$$|\mathbf{r} \times \mathbf{F}| = rF \sin \theta = (0.1)(5000 \sin \pi/2) = 500 \text{ Nm}$$