Essential Mathematics 2 – Introduction to the calculus

As you will already know, the calculus may be broadly separated into two major parts. The first part the *Differential Calculus* is concerned with finding the instantaneous rate of change of a function, so using it we are able, at any instant, to find how things change with respect to variables such as, time, distance or speed, especially when these changes are continually varying. The *Integral Calculus* has two primary functions, that of anti-differentiation, finding the *prime function* $f(x)$ from the *derived function* $f'(x)$ and second, that of summation, such as finding arc lengths, areas under graphs, surface areas, or volumes enclosed by a surface.

1 Notation

There are three common types of notation for representing the differential coefficient, or differential of a function. These are *Leibniz*, *functional* and *dot* notation.

Thus differentials using *Leibniz notation*, with which I am sure you will be familiar are represented as follows:

For the function $y(x)$, the first differential is $\frac{dy}{dx}$, the function y is the dependent variable and x is the independent variable. Do remember that the variables will differ according to the function being considered, thus, for example, the differential coefficient of the function $s(t)$ is $\frac{ds}{dt}$ which you may recognise as the rate of change of distance (s), with respect to time (t). The second and third differentials etc., in this notation are represented by $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, respectively. Leibniz notation is particularly appropriate for displaying the rules of differentiation.

Functional notation is particularly useful, when manipulating mathematical expressions for functions of differing variables. Thus in functional notation, the first, second, third derivatives etc., are represented by $f'(x)$, $f''(t)$, $f'''(g)$, where the *dash* is sometimes known as a *prime*.

Finally, particularly in the study of mechanics differentials may be represented by *dot notation*, e.g. $\dot{y}, \ddot{u}, \dddot{v}$ etc., where the variable is differentiated once, twice, three times, respectively.

Do remember that the differential of a function (its differential coefficient) is a measure of *the rate of change of a function*, which can be represented pictorially by the *slope* (*gradient*) of the graph of the function at a particular point.

2 Derivatives and differentiation

You will, it is hoped, be very familiar with being able to find the derivative of simple mathematical expressions, using the standard elementary rules. As a reminder and source of reference, the more common rules for some derivatives and their arithmetic combination, are set out below.

Note also, that the conditions for *maxima* and *minima* of a function are as follows.

$f(x)$ has a ***maximum*** value at $x = a$ if $f'(a) = 0$ and $f''(x)$ changes sign from *positive to negative* as x goes through the value of a ***or*** if $f'(a) = 0$ and $f''(a)$ is negative.

While $f(x)$ has a ***minimum*** value at $x = a$ if $f'(a) = 0$ and $f''(x)$ changes sign from *negative to positive* as x

Rules for some derivatives

Function $f(x)$	**Derivative $(f'(x))$**
ax^n	nax^{n-1}
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan(ax+b)$	$a\sec^2(ax+b)$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\log_e f(x)$	$\dfrac{f'(x)}{f(x)}$
Sum $\dfrac{d(u+v)}{dx}$	$\dfrac{du}{dx}+\dfrac{dv}{dx}$
Product $\dfrac{d(uv)}{dx}$	$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$
Quotient $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$	$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$
Function of a function or chain rule (if z is a function of x)	
$f(z)$	$\dfrac{df}{dz}\dfrac{dz}{dx}$

goes through the value of a ***or*** if $f'(a)=0$ and $f''(a)$ is *positive*.

Using these rules is a fairly straight-forward process, providing care is taken with the manipulation of the necessary algebra. *The golden rule is that you should, always attempt to simplify the function before trying to differentiate*. An attempt has been made within the Essential Mathematics sections of this book, to cover the necessary algebra, as it appears, in a sympathetic manner. However, if you still feel that you have a few weaknesses or lack the techniques needed to manipulate the algebra, you are advised to look first at the accompanying *Essential Mathematics* section on *Algebraic fundamentals*. If this is insufficient for your needs, then you might wish to refer to the book on *BTEC National Engineering* (third edition) by the same authors, where the fundamental algebra is covered in more detail.

In the first of the following examples, we practise the arithmetic of differentiation on a number of different elementary functions.

Example EM 2.1 Differentiate the following functions:

a) $f(x)=\left(\sqrt{x}\right)^3-\left(x^{-3}\right)^2$

b) $f(t)=\sin 4t-6\cos 2t+e^{-3t}$

c) $f(x)=\log_e(x^2+5)$

d) $y(x)=x^3\sin 2x$

e) $y(x)=\dfrac{e^{2x}}{x+3}$

f) $y(x)=\left(x^2-x\right)^9$

Then:

a) Simplifying the expression using the laws of indices gives, $f(x)=x^{3/2}-x^{-6}$ and so, $f'(x)=\dfrac{3}{2}x^{1/2}+6x^{-7}$

b) Applying the rules successively we get, $f'(t)=4\cos 4t+12\sin 2t-3e^{-3t}$

c) Again, just applying the appropriate rule, we get $f'(x)=\dfrac{2x}{x^2+5}$. Remember also that, $\log_e(x)=\ln(x)$

d) This function is a product, therefore,

$$\frac{dy}{dx}=(\sin 2x)(3x^2)+(x^3)(2\cos 2x)\text{ or,}$$
$$\frac{dy}{dx}=x^2(3\sin 2x+2x\cos 2x)$$

e) This function is a quotient, therefore,

$$\frac{dy}{dx}=\frac{(x+3)(2e^{2x})-(e^{2x})(1)}{(x+3)^2}\text{ and}$$
$$\frac{dy}{dx}=\frac{e^{2x}(2x+6-1)}{(x+3)^2}\text{ so}$$
$$\frac{dy}{dx}=\frac{e^{2x}(2x+5)}{(x+3)^2}$$

f) This function could be differentiated by using the basic rule but the bracketed expression would have to be expanded! Therefore, we will use the *function of a function rule*. Where if we let the bracketed expression $(x^2-x)=u$ so that $y=u^9$ and $\dfrac{dy}{du}=9u^8$ also $\dfrac{du}{dx}=2x-1$, then from

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

we get that,

$$\frac{dy}{dx} = (9u^8)(2x - 1)$$

and on substituting back our original expression for u, we get that,

$$\frac{dy}{dx} = 9(x^2 - x)^8(2x - 1).$$

In the next example we will consider one or two useful engineering applications of the differential calculus, involving *rate of change* problems.

Example EM 2.2

a) Suppose an empty spherical vessel is filled with water. As the water level rises, the radius of the water level in the vessel and the volume of the water will change. Now if the radius of water in the spherical vessel increases at 0.5 cm/s, find the rate of change of the volume of the water in the vessel, when the radius is 5 cm.

b) A particle is subject to harmonic motion given by the relationship, $x = A\sin\omega t$. Show that the linear acceleration is given by $a = -\omega^2 x$, where $x =$ the linear displacement from the centre of oscillation and $\omega =$ the angular velocity.

c) The motion of a body is modeled by the relationship, $s = t^3 - 3t^2 + 3t + 8$, where s is the distance in metres and t is the time in seconds. Find:

   i) The velocity of the body at the end of 3 seconds

   ii) The time when the body has zero velocity

   iii) Its acceleration at the end of 2 seconds

   iv) When its acceleration is zero.

Then

a) For a sphere $V = \frac{4}{3}\pi r^3$ and so, $\frac{dV}{dr} = 4\pi r^2$. Also the rate of change of the radius with time is given as $\frac{dr}{dt} = 0.5\,\text{m/s}$ and using the rule for a function of a function $\frac{dV}{dt} = \left(\frac{dV}{dr}\right)\left(\frac{dr}{dt}\right)$ then, $\frac{dV}{dt} = (4\pi r^2)(0.5) = 2\pi r^2$ and when, $r = 5$, $\frac{dV}{dt} = 50\pi = 157\,\text{cm}^3/\text{s}$.

b) Rate of change of distance with respect to time is velocity given by

$$v = \frac{dx}{dt} = A\omega\cos\omega t$$

and acceleration is rate of change of velocity with respect to time given by,

$$\frac{d^2x}{dt^2} = -A\omega^2\sin\omega t$$

but

$$x = A\sin\omega t$$

so

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

c) For (i) the velocity is given by,

$$\frac{ds}{dt} = 3t^2 - 6t + 3,$$

so at 3 seconds the velocity $= 3(3)^2 - 6(3) + 3 = 12\,\text{m/s}$.
For (ii) the body will have zero velocity when,

$$\frac{ds}{dt} = 3t^2 - 6t + 3 = 0,$$

solving this quadratic yields equal roots where $t = 1$, thus the body has zero velocity at time $t = 1$ second. For (iii) the acceleration is obtained by finding the rate of change of the velocity with respect to time, so differentiating a second time gives,

$$a = \frac{d^2s}{dt^2} = 6t - 6,$$

so when $t = 2$ seconds then, $a = 6\,\text{m/s}^2$. For part (iv) the acceleration is zero when, $a = 6t - 6 = 0$, i.e. when $t = 1$ second as expected.

In our final example, using the differential calculus, we look at turning points (TP) and consider whether or not they are a maximum, minimum or point of inflection. Remember that there will be a TP when the gradient/slope function is zero, i.e. when the differential for any function y, $\frac{dy}{dx} = 0$.

Example EM 2.3 For the function $y = x^3 - 3x$, find the TPs and determine their nature.

For TPs we require that $\frac{dy}{dx} = 3x^2 - 3 = 0$, so $x = \pm 1$. So to find the stationary values (SVs) corresponding to $x = \pm 1$, we substitute them into the original equation, $y = x^3 - 3x$ then, $y = (+1)^3 - 3(+1) = -2$ and $y = (-1)^3 - 3(-1) = +2$, so TPs at $(1, -2)$ and $(-1, +2)$. Now for point $(1, -2)$ we consider values above and below the value of $x = 1$, so choosing $x = 0$ and $x = 2$, and using the *gradient* equation $\frac{dy}{dx} = 3x^2 - 3$, when $x = 0$, $\frac{dy}{dx}$ is negative and when $x = 2$, $\frac{dy}{dx}$ is positive. So *gradient goes from negative to positive* so the point $(1, -2)$ is a *minimum*. Similarly at the point $(-1, 2)$ at values of $x = -2$ and $x = 0$ then when, $x = -2$, $\frac{dy}{dx} = 3x^2 - 3$, is positive and when $x = 0$, $\frac{dy}{dx} = 3x^2 - 3$, is negative, so *gradient goes from positive to negative* and the point $(-1, 2)$ is a *maximum*.

3 Integrals and integration

You should be familiar with integrating simple functions that involve most, if not all, of the rules tabulated below. The specific examples that follow this list have been designed to refresh your memory and to show one or two of the more important applications, directly related to engineering.

Rules for some integrals

y	$\int y dx$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$(ax+b)^n, n \neq -1$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$
$\frac{1}{x}(x > 0)$	$\ln x + C$
$\ln x$	$x \ln x - x + C$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b) + C$
$a^x (a\rangle 0)$	$\frac{a^x}{\ln a} + C$
$\frac{y'(x)}{y(x)}$	$\ln y(x) + C$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + C$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + C$
$\tan x$	$\sec^2 x + C$
$\tan(ax+b)$	$\frac{1}{a}\ln[\sec(ax+b)] + C$
$\sin^2 x$ or $\frac{1}{2}(1-\cos 2x)$	$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$
$\cos^2 x$ or $\frac{1}{2}(1+\cos 2x)$	$\frac{1}{2}x + \frac{1}{4}\sin 2x + C$
$\tan^2 x$ or $\sec^2 x - 1$	$\tan x - x + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin \frac{x}{a} + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan \frac{x}{a} + C$

Multiplication by a constant

If (k) is a constant then $\int ky(x)dx = k\int y(x)dx$

Sum rule

If $f(x)$ and $g(x)$ are functions capable of being integrated throughout the given range of x, then

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

Integration by parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$
$$\text{or } \int uv'dx = uv - \int vu'dx$$

Integration by substitution (or change of variable)

This rule is the integral equivalent of the function of a function rule used for differentiation and it is more easily memorised and understood using functional notation, as given here.

$$\int f(x)dx = \int f(g(x)) \times g'(x)dx$$

In this method a substitution such as, $u = g(x)$ is made and a simpler integral is compiled and solved in terms of the substituted variable (in this case u) and then once integrated, the original variable is substituted back to give the resulting integrand (the function after integration) in terms of the original variable, in this case (x).

Numerical integration

It is often the case that areas are not bounded by a perimeter that obeys a simple mathematical formula and so cannot be solved analytically. In this case we may need to resort to numerical integration to obtain a reasonable estimate. Two of the common methods for numerical integration are given here.

Simpson's rule: If a plane area is divided into a number of strips of equal width, then:

The area $= \dfrac{\text{common width}}{3} \times$ [sum of the first and last ordinates $+\, 4\times$ (the sum of the even ordinates) $+\, 2\times$ (the sum of the remaining ordinates)]

Trapezoidal rule: If a plane area is divided into strips of equal width, then:

The area = the common width × [half the sum of the first and last ordinates + the sum of the other ordinates]

Definite and indefinite integrals

When solving area problems, it will be necessary to know the boundary (limits) of the area under investigation and then use definite integrals to find the area concerned. *Definite integrals* are shown with the upper and lower limits attached to the integral sign, for example, $\int_b^a ydx$. Where, (a) is the upper limit and (b) the lower limit.

Integrals without limits are known as *indefinite integrals*, for example $\int ydx$.

There follow a number of examples that first show the method for using the rules and then show how we use the integral calculus to solve one or two specific problems, related to mechanical engineering.

Example EM 2.4 Integrate the following functions with respect to the given variables:

a) $\int (x^2 + 2x - 3)dx$

b) $\int (\sin 7\theta + 2\cos 5\theta)d\theta$

c) $\int (x+2)^{-1}dx \qquad x\rangle - 2$

d) $\int \left(e^{6t} - \dfrac{1}{e^{3t}}\right)dt$

e) Evaluate $\int_2^3 (1 + \cos 2\theta)d\theta$

f) Determine the shaded area between the function $y = 3x^2 + 10x - 8$ and the x-axis, as shown below (Figure EM 2.1).

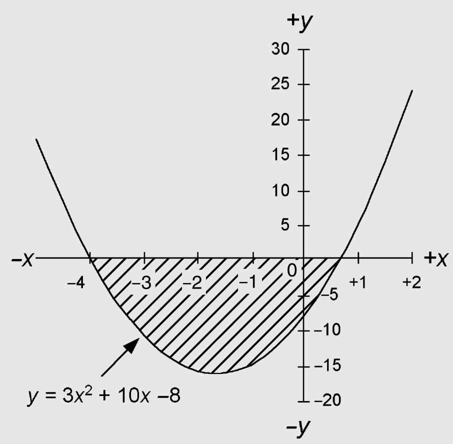

Figure EM 2.1 Curve of $y = 3x^2 + 10x - 8$

Then

a) Direct and successive use of the rule $\dfrac{x^{n+1}}{n+1} + C$ gives, $\dfrac{x^3}{3} + x^2 - 3x + C$

b) Following the rules for integrating sine and cosine functions gives

$$-\frac{1}{7}\cos 7\theta + \frac{2}{5}\sin 5\theta + C$$

c) The limits on x, allow integration using the Naperian log function so we get, $\ln(x+2)$

d) Simplifying using the laws of indices and then the rule for exponentials gives,

$$\int (e^{6t} - e^{-3t})dt = \frac{1}{6}e^{6t} + \frac{1}{3}e^{-3t} + C$$

e) For this definite integral we integrate first then evaluate between the limits, remembering that for trigonometric functions the angles are found in radian. Then,

$$\int_2^3 (1+\cos 2\theta)d\theta$$

$$= \left[\theta + \frac{1}{2}\sin 2\theta + C\right]_2^3 \text{ and so,}$$

$$= \left[3+\frac{1}{2}\sin(2)(3)+C\right] - \left[2+\frac{1}{2}\sin(2)(2)+C\right]$$

$$= \left[3+\frac{1}{2}\sin(6)+C\right] - \left[2+\frac{1}{2}\sin(4)+C\right] = 1.24$$

Note that the constants of integration are eliminated as a result of the subtraction between limits.

f) In order to find the required area we first need to find the limits of the integration, where the function crosses the x-axis, at $y=0$. Then $3x^2+10x-8=0$, factorizing we find that $x=\frac{2}{3}$ and $x=-4$. So to find the required area we integrate between these limits,

$$\int_0^{2/3} 3x^2 + 10x - 8dx$$

$$= \left[x^3 + 5x^2 - 8x + C\right]_0^{-4}$$

$$+ \left[x^3 + 5x^2 - 8x + C\right]_0^{2/3}$$

$$= \left[(-4)^3 + 5(-4)^2 - 8(-4) + C\right]_0^{-4}$$

$$+ \left[\left(\frac{2}{3}\right)3 + (5)\left(\frac{2}{3}\right)2 - (8)\left(\frac{2}{3}\right) + C\right]_0^{2/3}$$

$$= [(48+C) - (0+C)]$$

$$+ [(0.259+C) - (0+C)]$$

Then required area $= 48.259$ square units.

Example EM 2.5 Determine the following integrals using substitution or integration by parts, as indicated.

a) $\int \frac{1}{x^2+2x+1}dx$ by substitution

b) $\int \frac{x}{(x-3)^{1/2}\,dx}$ using the substitution, $u=(x-3)^{1/2}$

c) $\int x\sin x dx$ by parts

d) $\int x^2 \ln x dx$ by parts

Then

a) This integral requires a little algebraic manipulation to get it into a form to use a standard integral, the one we are going to use is the integral of the function $\frac{1}{a^2+x^2}$, so we need to get our integral in terms of this function, we may write our integral as $\int \frac{1}{(x+1)^2+1}dx$, by completing the square, I hope you can see this and that the integral takes the form we require, if we make the substitution $u=x+1$, so $\frac{dx}{du}=1$ and $du=dx$ then our integral becomes $\int \frac{1}{u^2+1}du$ and in this form we find that after integration we get $\arctan u + C$ and on substituting back, we get that,

$$\int \frac{1}{(x+1)^2+1}dx = \arctan(x+1) + C$$

b) In this case the substitution has been given in order to keep the algebra to a minimum and to make sure that after substitution the integral is more simply dealt with than before! So using, $u=(x-3)^{1/2}$ then,

$$\frac{du}{dx} = \frac{1}{2}(x-3)^{-1/2} = \frac{1}{2(x-1)^{1/2}}$$

therefore $\frac{dx}{du} = 2(x-3)^{1/2} = 2u$ and so $dx = 2udu$, also from the original substitution $u^2 = x-3$ and $x = u^2+3$. So that,

$$\int \frac{x}{(x-3)^{1/2}}dx = \int \frac{(u^2+3)}{u}2udu$$

$$= 2\int (u^2+3)du$$

then

$$\int \frac{x}{(x-3)^{1/2}}dx = 2\left[\left(\frac{u^3}{3}\right) + 3u\right] + C$$

or,

$$\int \frac{x}{(x-3)^{1/2}}dx$$
$$= \left[\frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2}\right] + C$$

c) Using the rule, $\int uv'dx = uv - \int vu'dx$ where we let $u = x$ and $v' = \sin x$ so $u' = 1$ and $v = -\cos x$, then

$$\int x\sin x dx = x(-\cos x) - \int(-\cos x)1dx \text{ and}$$
$$\int x\sin x dx = -x\cos x + \int \cos x dx, \text{ so that}$$
$$\int x\sin x dx = -x\cos x + \sin x + C$$

d) This is another product, hence the easiest way to solve the integral is by parts. In this case we will let $u = \ln x$, so $u' = \frac{1}{x}$, and $v' = x^2$, so $v = \frac{x^3}{3}$, then

$$\int x^2 \ln x dx$$
$$= (\ln x)\left(\frac{x^3}{3}\right) - \int\left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \text{ or}$$
$$\int x^2 \ln x dx$$
$$= (\ln x)\left(\frac{x^3}{3}\right) - \int\left(\frac{x^2}{3}\right)dx, \text{ so that}$$
$$\int x^2 \ln x dx$$
$$= \left(\frac{x^3}{3}\right)\ln x - \frac{x^3}{9} + C$$

In the final three examples we apply the integral calculus to the solution of problems concerned with the mechanics of solids.

Example EM 2.6 The differential equation (DE)

$$EI\frac{d^2y}{dx^2} = \left[R_A x - W_1\langle x-a\rangle - W_2\langle x-b\rangle\right]$$

is the Macaulay expression for the *bending moment* of a beam subject to bending under the action of point loads (W_1, W_2). All of the expression in the square brackets defines the bending moment (*BM* or just *M*) for the whole beam. The slope (gradient) of the bending is given by $\frac{dy}{dx}$ for the expression and the actual amount the beam is deflected is given by the expression for the deflection (y).

Assuming that all the symbols in the Macaulay expression are constants apart from the variables x and y, *integrate the DE once and obtain an expression for the slope, then integrate the expression a second time to obtain an expression for the deflection* (y).

Also, knowing that the constant of integration obtained from the second integration (B) = 0, obtain an expression for the constant of integration (A) resulting from the first integration of the expression, given that $x = l$ when the deflection $y = 0$.

Then

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[R_A x - W_1\langle x-a\rangle - W_2\langle x-b\rangle\right]$$

and integrating once gives,

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{R_A x^2}{2} - W_1\frac{\langle x-a\rangle^2}{2} - W_2\frac{\langle x-b\rangle^2}{2} + A\right]$$

and integrating a second time gives the expression for the deflection,

$$y = \frac{1}{EI}\left[\frac{R_A x^3}{6} - W_1\frac{\langle x-a\rangle^3}{6} - W_2\frac{\langle x-b\rangle^3}{6} + Ax + B\right]$$

Now applying the boundary conditions, where we are told that $B = 0$ and that $x = l$ when $y = 0$ so,

$$0 = EIy = \left[\frac{R_A x^3}{6} - W_1\frac{\langle x-a\rangle^3}{6} - W_2\frac{\langle x-b\rangle^3}{6} + Ax\right]$$

and

$$0 = \left[\frac{R_A l^3}{6} - W_1\frac{\langle l-a\rangle^3}{6} - W_2\frac{\langle l-b\rangle^3}{6} + Al\right]$$

or

$$A = \frac{R_A l^2}{6} - \frac{W_1(l-a)^3}{6l} - \frac{W_2(l-b)^3}{6l}$$

as required.

Example EM 2.7 Find an expression for the second moment of area of the rectangle shown, about its base edge.

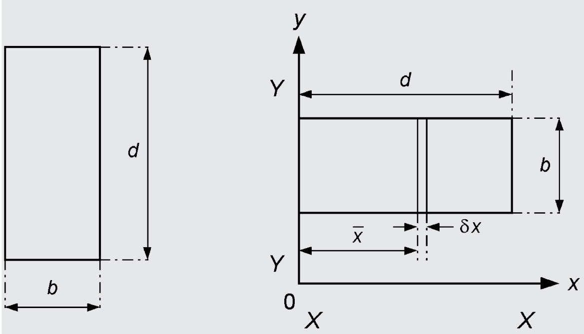

Figure EM 2.2 Set up for finding the second moment of area for a rectangle

The rectangle is shown set up on suitable axes. It is in this case convenient to turn the rectangle through 90° and let the base lie on the *y-axis* (YY). Figure EM 2.2 shows a typical element strip area parallel to the reference axis (YY), whose area is $b.\delta x$.

Now to find the second moment of area about the y-axis, we need to *sum all the areas of the elemental strips and multiply each of them by the distance* x^2 (second moment). Then the second moment of area

$$(I_{yy}) = \sum Ax^2 = \sum_{x=0}^{x=d} b\delta x(x^2) = \int_0^d bdx(x^2)$$

and so,

$$I_{yy} = b\int_0^d x^2 dx = b\left[\frac{x^3}{3}\right]_0^d = \frac{bd^3}{3}$$

In the final example concerning the integral calculus, you will see that it is mostly concerned with deriving expressions, from given data, having already gained an understanding of bending theory from studying Chapter 2.

Example EM 2.8 By considering the figure shown (Figure EM 2.3) of a small element of a beam carrying a uniformly distributed load, and the conditions required to maintain the beam in equilibrium, determine the following:

a) Show that the rate of shear $\frac{dF}{dx} = -\omega$

b) Show by neglecting the second-order terms involving $(\delta x)^2$ that, $\frac{dM}{dx} = F$

c) By considering the integrals of the results found in part a) and part b) show, $\omega = -\frac{d^2M}{dx^2}$.

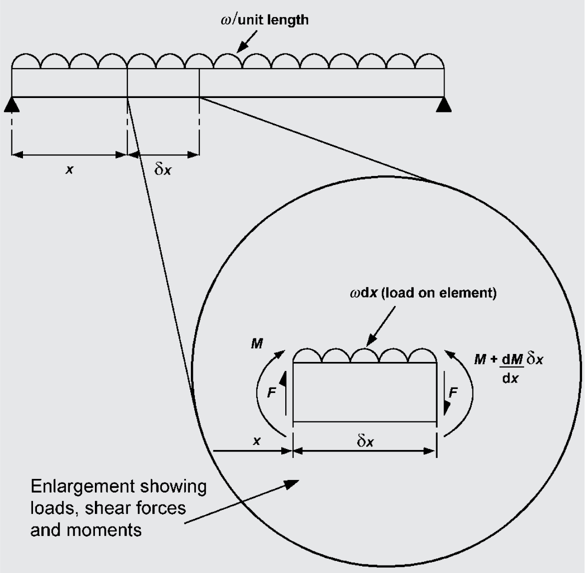

Figure EM 2.3 Small element from beam carrying a UDL

Then

a) From the figure we take a small element from the beam of length δx. Noting that the beam carries a distributed load ω that acts downwards. Then we can see that if F is the shear force due to the load at the point x, then at the point $x + \delta x$ the shear is $\frac{dF}{dx} \times \delta x$ and similarly if M is the bending moment at x, then $M + \frac{dM}{dx} \times \delta x$ is the bending moment at the point $x + \delta x$. Now for vertical equilibrium we have (adding downward forces), $-F + \omega\delta x + F + \frac{dF}{dx}\delta x = 0$, so that $\omega\delta x = -\frac{dF}{dx}\delta x$ or $\frac{dF}{dx} = -\omega$, as required,

b) To determine the required relationship we need to consider the tendency of the element to rotate, i.e. the equilibrium of moments (taken clockwise about the point x). Then:

$$M + \omega\delta x\frac{\delta x}{2} + \left(F + \frac{dF}{dx}\delta x\right)\delta x - \left(M + \frac{dM}{dx}\delta x\right) = 0$$

giving

$$M + \omega\delta x\frac{\delta x}{2} + F\delta x + \frac{dF}{dx}(\delta x)^2 - M - \frac{dM}{dx}\delta x = 0$$

and ignoring second-order terms in δx this gives, $F\delta x - \frac{dM}{dx}\delta x = 0$ or $\frac{dM}{dx} = F$, as required.

c) If we integrate the equation $\frac{dM}{dx} = F$ we get $M = \int F dx$ (this is saying that we can find values for the bending moment diagram by adding numerically the areas of the shear force diagram (i.e. integrating). Also from $\frac{dF}{dx} = -\omega$ we get that, $F = -\int \omega dx$ and substituting this into $\frac{dM}{dx} = F$ we get, $-\int \omega dx = \frac{dM}{dx}$ or $\omega = -\frac{d^2M}{dx^2}$ as required, indicating a distributed load acting downwards.