Essential Mathematics 3 – An introduction to differential equations

You have already met Differential Equations (DE) in *Essential Mathematics 2*, where you solved them using direct integration, here we will look at a few more very elementary techniques for solving some important first order and second order (DEs), that will be of use to you during your study of engineering science.

1 Introduction

A DE, as you have already seen, is one that contains differential coefficients. For example,

(1) $\frac{dy}{dx} = x + 3x^2 + \sin x$ or

(2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = -6y$ or

(3) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} = 0.$

Differential equations are classified according to the highest differential within the equation, thus equation (1) above is a *first-order differential equation*, while equation (2) is known as a *second-order differential equation* and equation (3) is a *third-order differential equation* and so on. We may solve DEs by integrating once, in the case of first-order DEs, twice in the case of second-order DEs and three-times in the case of third-order DEs and so on. If we find a solution where there is insufficient information provided to find the constant/s of integration we call this the *general solution*. If on the other hand we have been given sufficient *boundary conditions* that enable us to find specific values for the constants of integration, as you did when solving second-order bending DEs by direct integration (see Example 2.19) then we are able to find the *particular solution*, for the DE.

2 First-order differential equations

2.1 Solution by direct integration

The simplest type of first-order DEs are those that can be solved by integrating term-by-term. Thus all DEs of the form $\frac{dy}{dx} = f(x)$ are solvable by direct integration.

Example EM 3.1 If the velocity of a body is given by the DE $\frac{ds}{dt} = 3t^2 - 6t - 3$, where s = the distance in metres and t = time in seconds, find the *general solution* for the distance and the *particular solution* when $s = 4$, $t = 0$. Then by direct integration the general solution is, $s = t^3 + 3t^2 + 3t + C$ and applying the boundary conditions gives $C = 4$, so that the particular solution is $s = t^3 + 3t^2 + 3t + 4$.

Differential equations that take the form $\frac{dy}{dx} = ky$ may be solved by direct integration by recognising that this type of DE has a general solution $y = Ce^{kx}$ because the *exponential function* is the only function (say f), where $\frac{d}{dx}(f) = kf$, so it can always be defined in this way.

Example EM 3.2 Find the particular solution for $\frac{dy}{dx} = -3y$, given that $y = 2$ when $x = 0$. Then the *general solution* is given simply as $y = Ce^{-3x}$ and from the boundary conditions when $y = 2$ and $x = 0$ then, $2 = Ce^{0}$ or $C = 2$ and the particular solution is $y = 2e^{-3x}$.

2.2 Solution by separation of variables

Differential equations where there are two variables present, i.e. those that take the form $\frac{dy}{dx} = f(x)\bullet g(y)$, may be solved by rearrangement, whereby we separate each of the variables and then integrate them separately with respect to each other to obtain the solution, i.e. $\int \frac{dy}{g(y)} = \int f(x)dx$.

Example EM 3.3 If $3y^2\frac{dy}{dx} = 4x$ then

$$\int 3y^2 dy = \int 4x dx \quad \text{or} \quad y^3 = 2x^2 + C$$

is the general solution and if suitable boundary conditions are given, a particular solution may be found.

Example EM 3.4 Find the general solution for the equation $\frac{dy}{dx} = 3y - 6$ then $\left(\frac{1}{3y-6}\right)\frac{dy}{dx} = 1$ and so

$$\int \left(\frac{1}{3y-6}\right) dy = \int 1 dx,$$

$$\frac{1}{3}\ln(3y-6) = x + c$$

or

$$\ln(3y-6) = 3x + 3c$$

and after multiplication throughout by the Naperian logarithmic inverse (the exponential function) we get that $3y - 6 = e^{3x+3c}$ or $y = \frac{1}{3}e^{3x+3c} + 2$.

Example EM 3.5 Find the general solution for the equation

$$\frac{dy}{dx} = \frac{y^2}{x}$$

then on rearrangement

$$\frac{dy}{y^2} = \frac{dx}{x}$$

or

$$\frac{1}{y^2}dy = \frac{1}{x}dx$$

so integrating, i.e.

$$\int \frac{1}{y^2}dy = \int \frac{1}{x}dx,$$

gives

$$-\frac{1}{y} = \ln x + C$$

(now since C is a constant let it equal $\ln c$) then

$$-\frac{1}{y} = \ln x + \ln c$$

or from the laws of logs

$$-\frac{1}{y} \ln xc$$

or

$$y = \frac{1}{\ln xc}.$$

2.3 Solution by substitution

Equations of the form $P\frac{dy}{dx} = Q$, where P and Q are functions of x and y of the same degree, are known as *homogenous equations*. Thus for example the equation $\frac{dy}{dx} = x - y$ is homogenous since the terms are of the same degree in x and y, in this case degree one (x^1, y^1), while the equation $xy\frac{dy}{dx} = x^2 - y^2$ is also homogenous, since all the terms are of degree *two* in x and y, i.e. (x^1, y^1, x^2, y^2). Homogenous equations cannot be solved directly by separation of variables but if a *suitable substitution is used*, these equations can be put into a form where they may be solved by the separation of variables method. One such substitution is to use $y = zx$, where z is a function of x.

Example EM 3.6 So let us solve the homogenous equation

$$xy\frac{dy}{dx} = x^2 - y^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

using the substitution $y = zx$, from which we get (using the differentiation of a product rule)

$$\frac{dy}{dx} = (z)(1) + x\frac{dz}{dx},$$

then using this form of our substitution, the equation now becomes

$$z + x\frac{dz}{dx} = \frac{x^2 - x^2z^2}{x^2z}$$

or

$$z + x\frac{dz}{dx} = \frac{1 - z^2}{z}$$

and

$$z\left(z + x\frac{dz}{dx}\right) = 1 - z^2$$

so

$$zx\frac{dz}{dx} = 1 - 2z^2$$

on separation of variables we get that

$$\int \frac{z}{1 - 2z^2}dz = \int \frac{1}{x}dx.$$

Now if we can write the left-hand-side of this equation in a form so that the numerator is the derivative of the denominator, i.e. $\frac{f'(z)}{f(z)}$, then the integral is easily found as $f(z)$; look back at the rules of differentiation in *Essential Mathematics 2*. So multiplying both sides by -4 puts the LHS of the equation in the required form then

$$\int \frac{-4z}{1 - 2z^2}dz = \int \frac{-4}{x}dx$$

so after integration we get that

$$\ln(1 - 2z^2) = -4\ln x + C$$

and again if C is a constant we may let it equal $\ln C$ so that we have $\ln(1 - 2z^2) = -4\ln x + \ln C$ or we get that $1 - 2z^2 = Cx^{-4}$ (after multiplication throughout by the Naperian logarithmic inverse, the exponential function and after applying the laws of logarithms).

Now, substituting back for z gives $1 - 2\frac{y^2}{x^2} = Cx^{-4}$ or after multiplying every term by x^4 we get a general solution as $x^4 - 2x^2y^2 = C$.

2.4 Linear first-order differential equations

There is a set of DEs that can be put into the form $\frac{dy}{dx} + P(x)y = Q(x)$ where P and Q are functions of x only, that cannot be solved by any of the methods mentioned previously, these are known as *linear first-order differential equations* because y and its derivatives are all of the first degree. This type of DE may be solved using an integrating factor (IF), where $IF = e^{\int Pdx}$ this assumes that we can evaluate the integral of P. If we cannot then this method cannot be used. Each term in the equation must be multiplied by the IF, i.e.

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}P(x)y = e^{\int Pdx}Q(x) \qquad (1)$$

and remembering that the differential of the product $\frac{d}{dx}\left(e^{\int Pdx}y\right) = LHS$ of equation (1) we may write that $\frac{d}{dx}\left(e^{\int Pdx}y\right) = Q(x)e^{\int Pdx}$. The idea is that the IF will produce a 'perfect differential'. This differential (being a product) may be integrated by parts to solve the equation (see EM2 Example 2.5).

Example EM 3.7 Solve the equation

$$x^2\frac{dy}{dx} + xy = x^3.$$

We first place the equationin the correct form which is that given by

$$\frac{dy}{dx} + P(x)y = Q(x)$$

so for this example we get

$$\frac{dy}{dx} + \frac{y}{x} = x$$

(on division by x^2). We now identify the integrating factor, where we have $P(x) = 1/x$ so the

$IF = e^{\int \frac{1}{x} dx}$. Next we multiply each term of our equation by the IF to give

$$e^{\int \frac{dx}{x}} \frac{dy}{dx} + e^{\int \frac{dx}{x}} \frac{y}{x} = xe^{\int \frac{dx}{x}}$$

which may be written as

$$\frac{d}{dx}\left(e^{\int \frac{dx}{x}} y\right) = xe^{\int \frac{dx}{x}}$$

and integrating both sides gives

$$e^{\int \frac{dx}{x}} y = \int e^{\int \frac{dx}{x}} xdx \qquad (1)$$

and knowing that

$$\int \frac{dx}{x} = \ln x$$

then equation (1) becomes

$$e^{\ln x} y = \int e^{\ln x} xdx \quad \text{or} \quad xy = \int x^2 dx,$$

so that a general solution is

$$xy = \frac{x^3}{3} + C \qquad (2)$$

Now if we are given the boundary condition we are able to find a particular solution. Let $y = 1$ when $x = 3$. Then from equation (2) $3 = \frac{3^3}{3} + C$ giving $C = -6$ and a particular solution is $y = \frac{x^2}{3} - \frac{6}{x}$ (again from equation (2)).

3 Second-order differential equations

3.1 Solution of DEs with the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

Many practical problems in engineering may be modelled using *linear second-order differential equations* of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad (1)$$

where a, b, c are constant coefficients and $f(x)$ is a given function of x. A subset of this type of DE is found when $f(x) = 0$, i.e. when the equation takes the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (2)$$

A solution for equation (2) involving the *exponential function* may be found using what is known as, the *auxiliary equation*. We know that if

$$y = Ae^{mx}, \ \frac{dy}{dx} = Ame^{mx} \ \text{and} \ \frac{d^2y}{dx^2} = Am^2e^{mx}$$

substituting these values into equation (2) gives

$$a(Am^2e^{mx}) + b(Ame^{mx}) + c(Ae^{mx}) = 0$$

or

$$Ae^{mx}(am^2 + bm + c) = 0.$$

Therefore, $y = Ae^{mx}$ is a solution of equation (2) providing this *auxiliary equation*

$$(am^2 + bm + c) = 0.$$

This auxiliary equation will always be a *quadratic* for any second-order DE, where its roots for a *particular case* can be:

i) Real and different (real $m_1 \neq m_2$) giving the general solution to equation (2) as

$$y = Ae^{m_1x} + Be^{m_2x}.$$

ii) Real and equal ($m_1 = m_2$ or say m_1 twice) so that these two identical terms may be combined to give a single solution, say $y = Ce^{m_1x}$. However, every second-order DE has two arbitrary constants, so there must be one other term in the full solution. It can be shown that another solution is that $y = kxe^{m_1x}$, so we may add this to our single solution. Therefore a *complete* general solution to equation (2) under these circumstances is given by $y = e^{m_1x}(Ax + B)$ where again the letters A and B are used to represent the arbitrary constants.

iii) Complex (*say* $m = p \pm iq$) where it can be shown that the general solution to equation (2) under these circumstances is given by $e^{px}(A\sin qx + B\cos qx)$.

Example EM 3.8 Find the general solution for the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

and the particular solution when

$x = 0$, $y = 5$ and $\frac{dy}{dx} = 13$.

We first write down the auxiliary equation which is $m^2 - 5m + 6 = 0$ and find its roots. Then by factorisation we get that $(m-2)(m-3) = 0$ from which the roots are $m_1 = 2$ and $m_2 = 3$, so that from case (i) above we get our general solution as $y = Ae^{2x} + Be^{3x}$. Now for the particular solution we need to apply the boundary conditions. So applying the first set of conditions $x = 0$, $y = 5$ we get from the general solution that $5 = A + B$, also we get that

$$\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x}$$

and applying the boundary conditions $x = 0$, $\frac{dy}{dx} = 13$ we find that $13 = 2A + 3B$. So we now have two equations with two unknowns, so by solving them simultaneously, we get that $A = 2$ and $B = 3$. So that our particular solution will be $y = 2e^{2x} + 3e^{3x}$.

Example EM 3.9 Find the general solution for the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

Then from our DE the auxiliary equation is $m^2 + 6m + 9 = 0$ and again by factorisation we get that $(m+3)(m+3) = 0$. In this case the roots are the same, that is $m_1 = m_2 = -3$. So the general solution, from case (ii) above is found to be $y = e^{-3x}(Ax + B)$

A particular type of DE that takes the form

$$\frac{d^2y}{dx^2} + m^2y = 0 \qquad (3)$$

is known as a *harmonic equation* because it is used to model *simple harmonic motion* (SHM) that is *oscillatory* motion. The *solutions* for this type of equation are of the form

$$y = A\sin mx + B\cos mx \qquad (4)$$

since $\sin mx$ and $\cos mx$ are the only functions that have the property $\frac{d^2y}{dx^2} = -m^2y$. For example, if $y = \sin mx$, then $\frac{dy}{dx} = -m\cos mx$ and $\frac{d^2y}{dx^2} = -m^2\sin mx$ or $\frac{d^2y}{dx^2} + m^2y = 0$, which means that $y = \sin mx$ is a solution of equation (3).

Note also that the general solution to this type of equation is given by case (iii) above because any auxiliary equation formed from DEs of this type will produce complex roots. So, a complete general solution is given by $e^{px}(A\sin qx + B\cos qx)$.

Example EM 3.10 The oscillatory motion of a car suspension spring assembly is modelled by the SHM equation

$$\frac{d^2s}{dt^2} + 25s = 0.$$

If s is the displacement of the suspension spring from its equilibrium position after t seconds, determine this displacement in terms of the time, given that when $t = 0$, $s = 0$ and

$$\frac{ds}{dt} = 20.$$

Now the general solution can be found as before using the *auxiliary equation* method, which in this case is $m^2 + 25 = 0$ or $m^2 = -25$ so that $m = \pm i5$, i.e. the roots of the auxiliary equation are complex so that the general solution will take the form of $e^{px}(A\sin qx + B\cos qx)$, from case (iii) above, where in this case $p = 0$, the general solution will be

$$s = e^0(A\sin 5t + B\cos 5t)$$

or

$$s = A\sin 5t + B\cos 5t$$

which is the same general solution, as expected from equation (4).

Now to find the particular solution all we need do is apply the boundary conditions, since $t = 0$ when $s = 0$, then from our general solution

$$0 = B\cos 0,$$

so that $B = 0$. Also $t = 0$, when

$$\frac{ds}{dt} = 20 = 5A\cos 0$$

so that $A = 4$ and the particular solution is $s = 4\sin 5t$.

3.2 Solution of DEs with the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

It can be shown that a *complete general solution* of this type of equation can be achieved by first finding the general solution to the equation

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0 \qquad (5)$$

for u.

Where in the case of equation (5) u is called the *complementary function.* We have already found the general solution to equations of this type, where, these solutions are also the complementary function CF to equations of the type shown in equation (6), using the methods illustrated in Examples EM 3.8, 3.9 and 3.10.

What we need to do now, is find a particular solution for equations of the type

$$a\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = f(x) \qquad (6)$$

where v is the *particular integral* (PI) and *add it* to the CF found for equation (5). Then the complete general solution to equations in this form is

$$y = u + v = CF + PI \qquad (7)$$

The following example illustrates the whole procedure necessary to find the general solution to equations of this type.

Example EM 3.11 Obtain a general solution for the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2\sin 4x \qquad (8)$$

We need first to find the CF of equation (8) that is the general solution to the equation

$$\frac{d^2u}{dx^2} - 5\frac{du}{dx} + 6u = 0$$

Where, as before, we find the roots of the auxiliary equation, where in this case, the equation is

$$m^2 - 5x + 6 = 0 \text{ (see Example EM 3.8)},$$

therefore $m_1 = 2$ and $m_2 = 3$, so that from case (i) above we get our general solution (CF) as

$$u = Ae^{2x} + Be^{3x},$$

which encompasses both the constants of integration.

Now we need to find the PI for the equation

$$\frac{d^2v}{dx^2} - 5\frac{dv}{dx} + 6v = 2\sin 4x \qquad (9)$$

To find the PI, we need to do a little guess work that involves considering the form of $f(x)$. In this case we note that $f(x)$ involves the sine function and a good rule of thumb is that when $f(x)$ involves the sine or cosine functions we try the solution

$$v = C\sin ax + D\cos ax$$

or in this particular case we try

$$v = C\sin 4x + D\cos 4x.$$

Then

$$\frac{dv}{dx} = 4C\cos 4x - 4D\sin 4x$$

and

$$\frac{d^2v}{dx^2} = -16C\sin x - 16D\cos 4x$$

so on substituting these values into equation (9) we get

$$[-16C\sin 4x - 16D\cos 4x] - 5[4C\cos 4x - 4D\sin 4x]$$
$$+\ 6[C\sin 4x + D\cos 4x] = 2\sin 4x$$

and on simplifying this expression we get

$$-\ 10C\sin 4x - 10D\cos 4x - 20C\cos 4x$$
$$+\ 20d\sin 4x = 2\sin 4x.$$

Now equating the coefficients of the sine and cosine functions, respectively, we find that:

$$-\ 10C + 20D = 2$$
$$-\ 20C - 20D = 0$$

And solving these simultaneous equations we get that

$$C = -1/25 \text{ and } D = 2/25$$

so the *PI* for equation (9) is

$$v = -\frac{\sin 4x}{25} + \frac{2\sin 4x}{25}$$

and so the general solution to equation (8), from $y = u + v = CF + PI$ is

$$y = Ae^{2x} + Be^{3x} - \frac{\sin 4x}{25} + \frac{2\sin x}{25}$$

Now to enable a little of the guesswork to be taken out of finding a particular integral, there are a number of *PI*s that may be tried that are dependent on the nature of $f(x)$ some of these are listed below, for your convenience.

1) $f(x) =$ a constant try $v = k$
2) $f(x) = Ae^{ax}$ try $v = ke^{ax}$
3) $f(x) = a$ polynomial try $v = a + bx + cx^2 + \ldots$ as appropriate
4) $f(x) =$ the sine or cosine function try $v = C\sin ax + D\cos ax$

Note that if $f(x)$ contains an arithmetic combination of the above functions, a *PI* should be chosen that contains the same combination. So for example if $f(x)$ contains a sine or cosine function multiplied by an exponential function then a *PI* of the form $v = ke^{ax}(C\sin ax + D\cos ax)$, should be used.

You will only meet second-order differential equations, during your study of engineering science, where the *PI* and indeed, any other method of solution, will be fairly obvious. However, it is hoped, that the argument set out above will act as a suitable platform, for any further study of DEs.