Essential Mathematics 4: Algebraic Fundamentals

In this short extract we take a brief look at some of the algebraic fundamentals that will aid your understanding of much of the scientific analysis presented in the book. In particular we will consider:

- Factors and their theorems
- Partial fractions
- Indices and logarithms
- The Binomial theorem and Series.

1. Factors and Factor Theorems

1.1 ***Some useful common factors***

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

1.2 ***Factor Theorems***

Remainder theorem: When *f(x)* is divided by ($x - a$) the remainder is *f(a)and*
Factor theorem: If $f(a) = 0$, ($x - a$) is a factor of *f(x)*.
Repeated factors: If $(x-a)^2$ is a factor of *f(x)* then, $f(x) \equiv (x-a)^2 G(x)$ it can also be shown that when $(x-a)^2$ is a factor of *f(x)* then ($x - a$) is a factor of *f(x)* and of the derivative $f'(x)$ so that not only is $f(a) = 0$, but also $f'(a) = 0$.

The above theorems are useful for finding factors of a polynomial or for finding constants within polynomial expressions, as shown in Examples 1.1 and 1.2, that follow.

Example 1.1

Find the linear factors of $x^3 - 6x^2 + 3x + 10$. Now, the trick is to try the factors of the constant term that is $\pm 1,\ \pm 2,\ \pm 5$, then allowing $x^3 - 6x^2 + 3x + 10 = f(x)$ and remembering that for ($x - a$) to be a factor $f(a) = 0$, so trying the constant term factors we find that:

$f(1) = (1)^3 - 6(1)^2 + 3(1) + 10 \neq 0$
therefore $(x-1)$ is not a factor
$f(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10 = 0$
therefore $(x+1)$ is a factor
Similarly
$f(2) = 0$, so $(x-2)$ is a factor and
$f(5) = 0$, so $(x-5)$ is a factor
Now because *f(x)* is a polynomial to the power 3 there are only *three* linear factors and as the coefficient of x^3 is 1, then:
$x^3 - 6x^2 + 3x + 10 \equiv (x+1)(x-2)(x-5)$. Try multiplying up these factors to check the solution for yourself.

Example 1.2

Find the values of constants *A* and *B* if $(x+1)^2$ is a factor of $f(x) = 2x^4 + 7x^3 + 6x^2 + Ax + B$. Then following the argument for repeated factors after differentiating $f'(x) = 8x^3 + 21x^2 + 12x + A$ and since $(x+1)$ is a factor of *f(x) and* $f'(x)$,
then $f(-1) = 0$ and $f'(-1) = 0$ so that
$2(-1)^4 + 7(-1)^3 + 6(-1)^2 + A(-1) + B = 0$ or
$A - B = 0 \ldots.(1)$ and
$8(-1)^3 + 21(-1)^2 + 12(-1) + A = 0$ or $A = -1.....(2)$
so substituting equation (2) into (1), $B = -2$.

The technique shown in Example 1.2 is very similar to that required to find the constants in Macauley expressions, like that given in Example 2.18 in the book. To test your understanding try using the factor theorem to show that all the factors for *f(x)* given in Example 1.2 are,
$2(x+1)^2(x+2)(x-\tfrac{1}{2})$ or $(x+1)^2(x+2)(2x-1)$.

2. Partial Fractions

Relatively complex algebraic fractions can often be broken down into a series of smaller fractions which are much more easily manipulated these are known as *partial fractions*, the formulae given below show some typical standard relationships.

Partial fractions (*PF*s) are very useful when trying to integrate complex expressions that are able to be broken down in this way. They are also useful when determining the inverse of non-standard Laplace transforms, which cannot be identified from tables.

2.1 *Useful formulae*

Provided that the numerator *f(x)* is second degree or lower then:

$$\frac{f(x)}{(x+a)(x+b)(x+c)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)} \ldots(1)$$

$$\frac{f(x)}{(x+d)(ax^2+bx+c)} \equiv \frac{A}{(x+d)} + \frac{Bx+C}{(ax^2+bx+c)} \ldots\ldots(2)$$

$$\frac{f(x)}{(x+a)(x+b)^2} \equiv \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2} \ldots\ldots\ldots(3)$$

2.2 *Examples of the use of formulae*

Example 2.1

Find *PF*s for the expression $\frac{3x}{(x-1)(x-2)(x-3)}$.

This is an example of an expression that only contains linear factors, so using the format given by formula (1), we may write that;

$$\frac{3x}{(x-1)(x-2)(x-3)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}.$$

Now multiplying every term by $(x-1)(x-2)(x-3)$, that is the denominator and cancelling where appropriate gives;

$$3x \equiv A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Now we need to find values of the unknown `*x*' that satisfy the above equation, in order to determine the coefficients *A*, *B* and *C*.

There are two methods often used, each has its merits. For linear factors the following simple short-cut method will be used. You will meet the other method in Examples 2.2 and when we consider quadratic factors in Example 2.3.

In this case, all we need to do is substitute a value of '*x*' into the bracketed expressions which, when possible, reduces them to zero. For example the brackets multiplied by the coefficient *A* are $(x-2)(x-3)$ so choosing a value of $x = 2$ or $x = 3$ will reduce the whole expression to zero.

Whatever value of '*x*' is chosen, it must be applied to 'all *x*' throughout the whole expression.

So substituting $x = 2$ into the whole expression, gives $6 \equiv 0 + B(2-1)(2-3) + 0$ so that $B = -6$. Now repeating the process by substituting appropriate values, then when $x = 3$, we find that $9 \equiv 0 + 0 + C(3-1)(3-2)$ or $C = 4.5$ and when $x = 1$ we find that $3 \equiv A(1-2)(1-3) + 0 + 0$ or, $A = 1.5$, so that our expression in PF form is:

$$\frac{3x}{(x-1)(x-2)(x-3)} \equiv \frac{1.5}{(x-1)} - \frac{6}{(x-2)} + \frac{4.5}{(x-3)}.$$

Example 2.2

Determine the PFs for $\frac{1}{(x-2)^2(x-3)}$. In this case we have a *repeated* linear factor, which requires two PFs and so following formula 3, then we may write $\frac{1}{(x-2)^2(x-3)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)}$.

Now substituting appropriate values of x, as in our previous method would provide values of the coefficients very easily.

However, there is another method known as *equating coefficients*, which can be used independently, or in conjunction with method 1. Although rather more long-winded, it has the advantage that it can always be applied. Method 1 becomes rather more hit and miss when we deal with quadratic factors or higher order expressions. Here, we will combine method 1 with the new method of equating coefficients. So clearing fractions and then multiplying out the brackets we get

that $1 \equiv A(x-2)(x-3) + B(x-3) + C(x-2)^2$ or

$$1 \equiv A(x^2-5x+6) + B(x-3) + C(x^2-4x+4) \ldots\ldots(1)$$

substituting an appropriate value of *x*, as before say $x = 2$ we find that $B = -1$.

Now, we will equate coefficients. This requires us to set-up a very simple table containing the unknown in ascending *powers*, as follows.

From equation (1), the `*powers*' of the unknown involved are: x^0, x^1 and x^2. Noting that $x^0 = 1$ allows us to consider the constants in the equation, thus for example, $6x^0 = (6 \text{ x } 1) = 6$. So equating the coefficients of the powers of *x* in equation (1), gives:

$$x^0 \quad 1 = \ 6A - 3B + 4C \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x^1 \quad 0 = -5A + \ B - 4C \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$x^2 \quad 0 = \quad A \quad + C \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

So for equation (1) where we equate for the power x^0 we note that on the left hand side we have $1 = (1) \text{ x } (x^0)$, which is why it appears as a coefficient of x^0 in equation (1). Also multiplying the first expression $A(x^2-5x+6)$ to get $Ax^2 - 5Ax + 6A$, shows why 6A appears as the coefficient of x^0. This process is repeated for coefficients of x^1 and x^2 then, what we are left with is a set of linear simultaneous equations where we know the value of B. Then adding equation (1) to (2) gives, $1 = A - 2B$ and with $B = -1$ then $A = -1$ also from equation (3), $A = -C$ so that $C = 1$. Then:

$$\frac{1}{(x-2)^2(x-3)} \equiv \frac{1}{(x-2)} - \frac{1}{(x-1)^2} + \frac{1}{(x-3)}$$

Formula 2 shows the way in which an expression involving *linear and quadratic factors* should be split into *PF*s. To every quadratic factor like x^2+ax+b there corresponds, a *PF* $Cx+D/(x^2+ax+b)$. Repeated quadratic factors require additional *PF*s, thus a factor $(x^2+ax+b)^2$ would require *PF*s $Cx+D/(x^2+ax+b)$ and $Ex+F/(x^2+ax+b)^2$. Example 2.3, illustrates the method you will need to adopt when dealing with both linear and quadratic factors, in the denominator of an expression. The system of linear equations which results from equating coefficients is, in this example, fairly easily solved. More complex systems require the use of *matrix* methods for their solution.

Example 2.3

Express $\dfrac{x}{(x-1)^2(x^2+5x+1)}$ as a sum of partial fractions. Then

$$\frac{x}{(x-1)^2(x^2+5x+1)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+5x+1}$$

and multiplying every term by the LHS denominator and cancelling gives

$$x = A(x-1)(x^2+5x+1) + B(x^2+5x+1) + (Cx+D)(x-1)^2 \text{..........................}(1)$$

Using the method of equating coefficients of different powers of 'x' on *both* sides of the equation, gives:

$$\begin{aligned}
x^0:&\quad 0 = -A + B + D\\
x^1:&\quad 1 = (A-5A) + 5B + C + 2D\\
x^2:&\quad 0 = 4A + B - 2C + D\\
x^3:&\quad 0 = A + C
\end{aligned}$$

Make sure you see how these values were obtained. If you find this difficult multiply out the brackets in Equation (1), and try again. Now we need to solve this system of linear equations to find coefficients, in this case, this is relatively easy. Numbering the equations then

$$\begin{aligned}
x^0:&\quad 0 = -A + B + D \quad (1)\\
x^1:&\quad 1 = -4A + 5B + C + 2D \quad (2)\\
x^2:&\quad 0 = 4A + B - 2C + D \quad (3)\\
x^3:&\quad 0 = A + C \quad (4)
\end{aligned}$$

Starting from the simplest equation we see from (4) that $A=-C$ and substituting for C in equations (2) and (3) we get

$$\begin{aligned}
1 &= -5A + 5B + 2D \quad (2')\\
0 &= 6A + B + D \quad (3')
\end{aligned}$$

Now subtracting (1) from (3′)

$$\begin{aligned}
&0 = 6A + B + D\\
&-(0 = -A + B + D)\\
&0 = 7A
\end{aligned}$$

This implies that $A=0$. Now substituting this value of A into $(2')$ and $(3')$ gives

$$\begin{aligned}
1 &= 5B + 2D\\
0 &= B + D
\end{aligned}$$

And solving simultaneously $1=7B$ so $B=1/7$ and so $D=-1/7$, then required PFs are:

$$\frac{x}{(x-1)^2(x^2+5x+1)} = \frac{1}{7(x-1)^2} - \frac{1}{7(x^2+5x+1)}.$$

Note that we could have found B initially using method (1) this would have simplified the manipulation of the equations.

If the numerator of an expression is of higher degree than the denominator then we have an improper fraction. To convert an improper fraction to a proper fraction, we need to use long division of algebra. You should consult an appropriate A level or National level mathematical text, if you are unfamiliar with this technique.

3. Indices and Logarithms

3.1 Indices

Laws of indices:

In the following laws 'a', is the common *base*, 'm' *and n* are the *indices* (powers/exponents). Each law has an example of its use, alongside.

1) $a^m \times a^n = a^{m+n}$ $\quad 2^2 \times 2^4 = 2^{2+4} = 2^6 = 64$

2) $\dfrac{a^m}{a^n} = a^{m-n}$ $\quad \dfrac{3^4}{3^2} = 3^{4-2} = 3^2 = 9$

3) $(a^m)^n = a^{mn}$ $\quad (2^2)^3 = 2^{2\times 3} = 2^6 = 64$

4) $a^0 = 1$ (Any number raised to the power 0 is always 1)

5) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $\quad 27^{\frac{4}{3}} = \sqrt[3]{27^4} = 3^4 = 81$

6) $a^{-n} = \dfrac{1}{a^n}$ $\quad 6^{-2} = \dfrac{1}{6^2} = \dfrac{1}{36}$

Example 3.1

Evaluate the following expressions:

$a)\ \frac{3^2 \text{ x } 3^2 \text{ x } 3}{3^4}$, $b)\ (6)(2x^0)$, $c)\ 36^{-\frac{1}{2}}$, $d)\ 16^{-\frac{3}{4}}$,

$e)\ \frac{(2^3)^2(3^2)^3}{3^4}$.

$$a)\ \frac{3^2 \text{ x } 3^2 \text{ x } 3}{3^4} = \frac{3^{2+3+1}}{3^4}(\text{law 1}) = \frac{3^6}{3^4} = 3^{6-4}(\text{law 2})$$
$$= 3^2 = 9$$
$$b)\ (6)(2x^0) = (6)(2)\ (\text{law 4}) = 12$$

$$c)\ 36^{-\frac{1}{2}} = \frac{1}{36^{\frac{1}{2}}}\ (\text{law 6}) = \frac{1}{\sqrt{36}}\ (\text{law 5}) = \pm\frac{1}{6}$$

$$d)\ 16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}}\ (\text{law 6}) = \frac{1}{\sqrt[4]{16^3}}\ (\text{law 5}) = \frac{1}{2^3} = \frac{1}{8}$$

$$e)\ \frac{(2^3)^2(3^2)^3}{3^4} = \frac{(2^{3\text{x}2})(3^{2\text{x}3})}{3^4}\ (\text{law 3}) = \frac{2^6 \text{ x } 3^6}{3^4} =$$
$$2^6 \text{ x } 3^{6-4}\ (\text{law 2}) = 2^6 \text{ x } 3^2 = 64 \text{ x } 9 = 576$$

Example 3.2

Simplify the following expressions

$a)\ \frac{12x^3y^2}{4x^2y}$, $b)\ \frac{a^3b^2c^4a^2}{a^4bc^3}$ $c)\ \left[b^3c^2ab^3c^2a^0\right]$

$$a)\ \frac{12x^3y^2}{4x^2y} = 3x^{3-2}y^{2-1} = 3xy$$

(law 2 and division of integers)

$$b)\ \frac{a^3b^2c^4a^2}{a^4bc^3} = a^{3+2-4}b^{2-1}c^{4-1-2} = abc$$

(law 2 and operating on like bases)

$$c)\ \left[b^3c^2ab^3c^2a^0\right]^2 = \left[b^3c^2ab^3c^2(1)\right]^2\ (\text{law 4})$$
$$= \left[ab^{3+3}c^{2+2}\right]^2\ (\text{law 1}) = \left[ab^6c^4\right]^2$$
$$= a^2b^{12}c^8\ (\text{law 3}).$$

3.2 Logarithms and logarithmic functions

Laws of logarithms

1) If $a = b^c$, then $c = \log_b a$

2) $\log_a MN = \log_a M + \log_a N$

3) $\log_a \frac{M}{N} = \log_a M - \log_a N$

4) $\log_a \left(M^n\right) = n\log_a M$

5) $\log_b M = \frac{\log_a M}{\log_a b}$

Exponential and logarithmic functions

A number in *exponent* or *index form* may be written as $y = a^x$ (see the laws of indices).

The *exponential function* is written as $\exp x = e^x$.

The natural or *Naperian logarithm* of some unknown value '*x*' is written as $\log_e x = lnx$.

The *logarithmic function* to *any base* (a) is written as $y = \log_a x$.

Manipulation of logarithmic functions and formulae where the variable required is part of an index requires the use of the laws of logarithms to find the variable. For example, if $a = b^c$ and we need to make 'c' the subject of the formula, then from laws (1) and (4) and taking logarithms to base 'b' gives $\log_b a = c\log_b b$ and as $\log_b b = 1$ then $c = \log_b a$, as required.

Example 3.3

If $U_2 = U_1e^{\left(w/pv\right)}$ make w the subject of the formula.

$\log_e\left(\frac{U_2}{U_1}\right) = \left(\frac{w}{pv}\right)\log_e e$ (after division by U_1 and applying logarithms) and because $\log_e e = 1$ then,

$\log_e\left(\frac{U_2}{U_1}\right) = \frac{w}{pv}$ therefore $w = pv\log_e\left(\frac{U_2}{U_1}\right)$

Manipulating logarithmic equations or expressions to find a new subject often require the use of the *inverse function*. So, you should be aware that the inverse of the exponential function (e^x) is $\log_e x$ or $\ln x$ (as on your calculator). Also that, the inverse of the function 10^x is $\log_{10} x$.

Example 3.4

Transpose $b = \log_e t - a\log_e D$ (for t) then:

$b = \log_e t - \log_e D^a$ (law 4)

$b = \log_e\left(\frac{t}{D^a}\right)$ (law 3)

$e^b = \frac{t}{D^a}$ (after applying the inverse function e^x of $\log_e x = \ln x$ to both sides), therefore

$t = e^b D^a$

The next example involves the solution of an indicial equation that requires a little algebraic manipulation.

Example 3.5

Solve the equation $2(2^{2x}) - 5(2^x) + 2 = 0$ for x.

The trick is to simplify this equation using a substitution and in performing the substitution we need to be aware of the laws of indices.

If we substitute $y = 2^x$, (where

$2^{2x} = (2^x)(2^x) = (y \text{ x } y) = y^2$), then we get that

$2y^2 - 5y + 2 = 0$ ……………………….(1)

Now equation (1) is a quadratic that may be solved simply by factorisation.

Then $(2y-1)(y-2) = 0$ from which

$2y - 1 = 0$ giving $y = \frac{1}{2}$ or $y - 2 = 0$ giving $y = 2$.

Therefore $2^x = \frac{1}{2}$ or $2^x = 2$, then by inspection,

$x = \pm 1$.

If you could not see this straightaway, then note, from the laws of indices that $a^{-1} = \frac{1}{a^1}$ and $a^1 = a$, where in this case of course $a = 2$.

The final example associated with the manipulation of logarithmic functions, involves a power equation for belt tension in a transmission system (see belt drives on pages 184 – 186 of the book).

Example 3.6

The formula $P = T\left(1 - e^{-\mu\theta}\right)v$, relates the power ($P$), belt tension ($T$), angle of lap (θ), linear velocity (v) and coefficient of friction (μ) for a flat belt drive system.

Determine an expression for the coefficient of friction and determine its value when,

$P = 2500,\ T = 1200,\ v = 3$ and $\theta = 2.94$.

Then from formula, $-e^{-\mu\theta} = \frac{P}{Tv} - 1$ or $e^{-\mu\theta} = 1 - \frac{P}{Tv}$

and taking logarithms (to the base e) of both sides

gives $-\mu\theta = \ln\left(1 - \frac{P}{Tv}\right)$ or $\mu = -\ln\left(1 - \frac{P}{Tv}\right)\left(\theta^{-1}\right)$.

Now, substituting the given values, we get

that $\mu = -\ln\left(1 - \frac{2500}{(1200)(3)}\right)\left(\frac{1}{2.94}\right)$

or $\mu = -\ln(0.306)\left(\frac{1}{2.94}\right) = -(-1.18417)(0.340)$

and so the coefficient of friction $\mu = 0.4$.

4. Series and the Binomial Theorem

4.1 Notation

The terms of a *sequence* when added form a series. For example the set of numbers; 1, 2, 4, 8, 16 are in a set order and form a sequence because each number may be found from the previous number by applying an obvious rule. When this sequence is *added* a series is formed.

To define a series more concisely we use *sigma notation*, where the Greek letter sigma $\sum =$ *the sum of all the terms*. The range of summation depends on the highest and lowest terms positioned above and below the sigma sign, respectively.

Thus:

$\sum_{n=1}^{n=\infty} a^n =$ the sum of all terms a^n between $n = 1$ and $n = \infty$ (an infinite *unbounded* series that is a series with no *upper bound*)

This series can be *bounded* by stipulating the number of terms in the series for example $\sum_{n=1}^{n=5} 2^n$ and writing this out in full we find that:

$$\sum_{n=1}^{n=5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$

$$= 1 + 2 + 4 + 8 + 16 + 32$$

Consider the following series $\sum_{n=1}^{n=\infty} \frac{1}{2^n}$ does this series have a *limit*? In other words does it have an upper and lower boundary? Expanding the series we get:

$$\sum_{n=1}^{n=\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} \ldots\ldots \frac{1}{2n}$$

$$\sum_{n=1}^{n=\infty} \frac{1}{2^n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots\ldots \frac{1}{2n}$$

If we sum just the first term of the series the lower bound will be 1. Also as n approaches ∞ then $\frac{1}{2^n}$ approaches 0, there must also be an upper bound because successive terms approach zero.

The above example is a ***geometric series*** because each successive term is obtained from the preceding term by multiplication of a common ratio (r) which for this series is $r = \frac{1}{2}$. This is a *convergent series* because the common ratio r is less than 1. If a geometric series is convergent then an upper bound can be found from the relationship $S_\infty = \frac{a}{1-r}$ where the positive value of $|r| < 1$ and $a =$ the first term. Then for the above series $S_\infty = \frac{1}{1-\frac{1}{2}} = 2$ and so has a *limit*.

Example 4.1

Find the upper and lower limits of x so that the series $\sum_{n=1}^{n=\infty} \frac{(x-1)^n}{2^n}$ converges and determine the sum to infinity of the series when $x = 2$. Then by expanding the series $\sum_{n=1}^{n=\infty} \frac{(x-1)^n}{2^n} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3}$, it can be seen that the common ratio is $r = \left(\frac{x-1}{2}\right)$.

Applying the condition for convergence gives us in this case $\left|\frac{x-1}{2}\right| < 1$ and removing the modulus sign allows us to consider the negative limit, i.e.

$-1 < \frac{x-1}{2} < 1$ which on multiplication by 2 gives:

$-2 < x-1 < 2$ so upper limit from $x-1 < 2$ is $x < 3$.

So, x must lie between a lower limit greater than -1 and an upper limit less than 3.

Therefore for $x = 2$, the series will converge and its upper limit (sum to infinity) is found sing $S_\infty = \frac{a}{1-r}$, where from $\left|\frac{x-1}{2}\right| < 1$, when $x = 2$, we find that $S_\infty = \frac{1}{\left(1-\frac{1}{2}\right)} = 2$.

Note that if we put $x = 2$, into the series $\sum_{n=1}^{n=\infty} \frac{(x-1)^n}{2^n} = 1 + \frac{x-1}{2} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3}$, we obtain the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.....*etc.*, as we did before for the series $\sum_{n=1}^{n=\infty} \frac{1}{2^n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots \frac{1}{2n}$.

We now need to mention one other type of series before considering power series and hyperbolic functions. The series $a + (a+d) + (a+2d) + \ldots.$, in which each term differs from the previous one by the same amount d, is called an ***arithmetic series*** or progression. The *nth* term in the series is $a + (n-1)d$ and the *sum of the first n terms* is $S_n = \frac{1}{2}n[2a + (n-1)d]$, or if the *first term* (a) and *last term* (l) are given but *not* the common difference the sum is given by $S_n = \frac{1}{2}n(a+l)$.

Example 4.2

How many terms in the arithmetic series 2, 3.5, 5, must be taken if the sum is to be 123?

We know that $a = 2$ and $d = 1.5$, so using $\frac{1}{2}n[2a + (n-1)d]$, we have that

$123 = \frac{1}{2}n[(2)(2) + (n-1)(1.5)]$ and on simplification

$123 = \frac{1}{2}n[1.5n + 2.5]$ so that $3n^2 + 5n - 492 = 0$.

Now this quadratic factorises to give $(n-12)(3n+41) = 0$ from which we find that $n = 12$ or $n = -41/3$ and since (n) is a number of terms, it must be positive, so it will take 12 terms to ensure that the sum is 123.

4.2 Useful formulae for the Binomial theorem, series and hyperbolic functions

Binomial theorem

$${}^nP_r = \frac{n!}{(n-r)!} \quad \text{permutations} \quad (1)$$

$${}^nC_r = \frac{n!}{r!(n-r)!} \quad \text{combinations} \quad (2)$$

$$(a \pm b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots \quad (3)$$

$$(a+b)^n = a^2 + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots (4)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + (5)$$

Series

$e^x = \exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$(6) Valid for 'all x'

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$(7) Valid for $-1 < x \le 1$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$(8) Valid for $-1 \le x < 1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$(9) Valid for 'all x', x in radian

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots$(10) Valid for 'all x', x in radian

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$(11)

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots$(12)

Hyperbolic functions

$e^x = \cosh x + \sinh x$ (13)

$e^{-x} = \cosh x - \sinh x$ (14)

$\sinh x = \frac{e^x - e^{-x}}{2}$ (15)

$\cosh x = \frac{e^x + e^{-x}}{2}$ (16)

$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$ (17)

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ (for all x) (18)

$\cosh^{-1} x = \pm\ln\left(x + \sqrt{x^2+1}\right)$ ($|x| > 1$) (19)

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ (20)

4.3 Binomial theorem

Formula (2) tells us how many combinations there are of selecting (r) objects from (n) objects. For *combinations* the selection is made *irrespective of order*, for *permutations* formula (1) enables us to calculate the selection of (r) objects from (n) *in a specified order*. For the binomial theorem we are primarily interested in combinations, as in formulae (3) to (5). So for example, the selection of four objects from six, irrespective of order is given by formula (2) as

$$^6C_4 = \frac{n!}{r!(n-r)!} = \frac{6!}{4!(6-4)!} = \frac{6\text{x}5\text{x}4\text{x}3\text{x}2\text{x}1}{(4\text{x}3\text{x}2\text{x}1)(2\text{x}1)} = 15$$

Note that a number followed by an exclamation mark (!) is a *factorial* number that is a number which multiplies itself by all the numerals down to one, as shown in the above example.

Example 4.3

Expand $(1+2x)^{-1/2}$ to 4 terms. Then using formula (5), we find that:

$$(1+2x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)(2x) + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}-1\right)(2x)^2}{2!} + \frac{\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}-1\right)\left(-\tfrac{1}{2}-2\right)(2x)^3}{3!} + \ldots.$$

$$= 1 - x + \tfrac{3}{2}x^2 - \tfrac{5}{2}x^3 + \ldots.$$

restriction $-\frac{1}{2} < x < +\frac{1}{2}$ for convergence.

Example 4.4

Expand $(x-y)^5$ using the binomial theorem, then from formula 4, we find that:

$$(x-y)^5 = x^5 + (5)(x)^4(-y)^1 + \frac{(5)(4)(x)^3(-y)^2}{2!} + \frac{(5)(4)(3)(x)^2(-y)^3}{3!} + \frac{(5)(4)(3)(2)(x)^1(-y)^4}{4!} + \frac{(5)(4)(3)(2)(1)(-y)^5}{5!}$$

And simplifying gives,

$$(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

From example 4.3 and formula (5), it can be seen that a series can be produced for functions of the type $(1+x)^n$, where n is a whole, fractional or negative number, by using the binomial theorem to the required degree of accuracy. It is this use of the binomial, where we are able to find *approximations* for algebraic expressions in the form of a *power series*, that we consider next.

4.4 Series

Using the binomial expansion we can derive particularly useful series such as the *exponential series* (see example 4.5) and also determine *estimates* of exponential, logarithmic and hyperbolic functions using *power series* (i.e. a series where each term is a simple power of the independent variable).

Example 4.5

Expand $\left(1+\frac{x}{n}\right)^n$ using the binomial theorem and deduce what happens to the series as $n \to \infty$. Then using formula (5), we find that:

$$\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{n(n-1)}{2!}\left(\frac{1}{n}x\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}x\right)^3 + ...$$

$$= 1 + x + \frac{\left(1-\frac{1}{n}\right)}{2!}x^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}x^3 + ...$$

Now we may deduce that for the terms $\frac{1}{n}, \frac{2}{n}$, etc. as $n \to \infty$, these terms $\to 0$ therefore the bracketed expressions, all approach 1, or *in the limit* equal 1.

So in the limit

when $n \to \infty$, $\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

I hope you recognise this series as the exponential series that is formula (6), where:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

This series provides us with *the base* for natural or Naperian logarithms $(\ln \text{ or } \log_e)$, when $x = 1$. So that, $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ...$

$$= 1 + 1 + 0.5 + 0.16666 + 0.041666 + 0.008333$$

Then, e^1 □ 2.72 correct to 2 decimal places. Continuing the series gives us a better and better approximation for e, where correct to five decimal places e □ 2.71828.

Example 4.6

Determine the value of $\cosh(1.932)$ using the definition given in formula (16) for hyperbolic functions and compare this with the approximation obtained from the first four terms of its power series, formula (12).

$\cosh x = \frac{e^x + e^{-x}}{2}$ so $\cosh(1.932) = \frac{e^{1.932} + e^{-1.932}}{2}$ and letting $y = e^{1.932}$ then, $\log_e y = 1.932$ so $y = 6.9033$.

Also $\frac{1}{y} = e^{-1.932} = 0.1449$ so,

$\cosh x = \frac{6.9033 + 0.1449}{2} = 3.524$ (3 decimal places).

Now using equation (12), $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

$$\cosh(1.932) = 1 + \frac{(1.932)^2}{2} + \frac{(1.932)^4}{24} + \frac{(1.932)^6}{720} + ..$$

$$= 1 + \frac{3.7326}{2} + \frac{13.9325}{24} + \frac{52.0047}{720}$$

$$= 1 + 1.8663 + 0.58055 + 0.0722$$

$$= 3.52$$

An accuracy of three decimal places would require a minimum of five terms.

In this final example of the use of the Binomial theorem and power series we consider the simplification of a formula concerned with the analysis of mechanisms (see section 9.2.2 starting on page 176 of the book).

Example 4.7

The formula for the displacement (s) of a slider crank may be written as

$s = r\cos\theta + l\sqrt{(1 - n^2\sin^2\theta)}$ show using the binomial theorem that a good *approximation* for the displacement is

$s = r\cos\theta + l\left(1 - \frac{n^2}{2}\sin 2\theta\right)$ when $n = 1/2$.

Then, considering the expression under the square root sign, using equation (5)

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$ and

letting, $x = n^2\sin^2\theta$ we find that:

$$\left(1 - n^2\sin^2\theta\right)^{1/2} = 1 + \left(\tfrac{1}{2}\right)\left(-n^2\sin^2\theta\right) + \frac{\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)}{1 \text{ x } 2}\left(-n^2\sin^2\theta\right)^2 + \frac{\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}{1 \text{ x } 2 \text{ x } 3}\left(-n^2\sin^2\theta\right)^3 + ...$$

$$\left(1 - n^2\sin^2\theta\right)^{1/2} = 1 + \frac{n^2}{2}\sin^2\theta - \frac{n^4}{2}\sin^4\theta - \frac{n^6}{2}\sin^6\theta ...$$

Now, because $n = 1/2$ and $\sin\theta \le 1$, the terms in the above expression will quickly converge towards zero. Therefore a good approximation may be obtained by considering only the first two terms of the expansion, i.e. $\left(1 - n^2\sin^2\theta\right)^{1/2} \approx 1 + \frac{n^2}{2}\sin^2\theta$.

Then, a good approximation for the displacement of the slider will be $s = r\cos\theta + l\left(1 - \frac{n^2}{2}\sin 2\theta\right)$ as required.

Footnote: If you have found difficulty with any of the algebra presented in this short extract you are strongly recommended to revise the elementary background algebra found in *BTEC National Engineering* by Tooley, M and Dingle, L.