Essential Mathematics – Trigonometric Formulae and Identities

1. Some Fundamental Formulae

1. Radian measure

1.1 $\pi rad = 180^{\circ}$
1.2 If θ is small and measured in radian, then
$\sin\theta = \theta$, $\cos\theta = 1$ and $\tan\theta = \theta$
1.3 arc length $s = r\theta$ (θ in radian)
1.4 area of sector $= \frac{1}{2}r^2\theta$ (θ in radian)

2. Triangle

2.1 Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
2.2 Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$
2.3 Area $= \frac{1}{2}bc\sin A$
$= \sqrt{s(s-a)(s-b)(s-c)}$
where, $s = \frac{(a+b+c)}{2}$

3. Polar and Cartesian coordinates

3.1 $x = r\cos\theta$, $y = r\sin\theta$
3.2 $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left[\frac{y}{x}\right]$, $\tan\theta = \frac{y}{x}$

4. Equations of circle

4.1 Centred at origin, radius (a), $x^2 + y^2 = a^2$
4.2 General equation, $x^2 + y^2 + 2gx + 2fy + c = 0$
Centred at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

5. Superposition

5.1 $a\sin\omega x + b\sin\omega x = c\sin(\omega x + \phi)$ where point (a, b) has polar coordinates r, θ, $c = r = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$

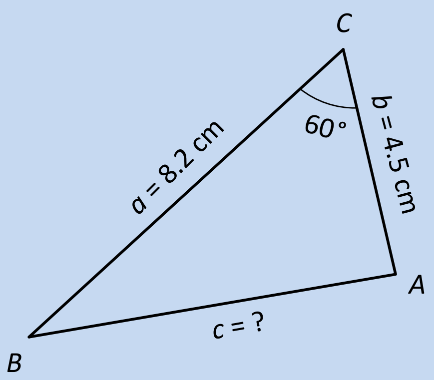
There follows some examples of the use of the above formulae.

Example 1.1

An arc AB of length 4.5 cm is marked on a circle of radius 3 cm. Find the area of the sector bounded by this arc and centre of the circle.
Then from formulae 1.3 and 1.4, the arc angle is $\theta = \frac{s}{r} = \frac{4.5}{3} = 1.5rad$ and so the area of the sector is $= \frac{1}{2}r^2\theta = (0.5)(3^2)(1.5) = 6.75cm^2$

Example 1.2

In the triangle ABC, $BC = 8.2cm$, $AC = 4.5cm$ and $\angle ACB = 60^{\circ}$. Calculate:
(i) The other angles of the triangle,
(ii) The area of the triangle.



First we sketch the triangle to establish the sides a, b and c.
Then using the cosine rule (formula 2.2),
$c^2 = a^2 + b^2 - 2ab\cos C$ we find that
$c^2 = (8.2)^2 + (4.5)^2 - 2(8.2)(4.5)\cos 60$
$c^2 = 67.24 + 20.25 - 36.9$
$c = 7.11cm$

The question asks us to find all the other angles of the triangle, so using the sine rule
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ we find that
$\frac{8.2}{\sin A} = \frac{4.5}{\sin B} = \frac{7.11}{\sin C}$ and from the last two terms
$\sin B = \frac{4.5\sin 60}{7.11} = 0.3169$ and $\angle B = 18.5^{\circ}$
therefore $\angle A = 101.5^{\circ}$

The area of the triangle may now be found using formula 2.3, Area $= \frac{1}{2}ab\sin C$ therefore

$Area = (0.5)(8.2)(4.5)\sin 60 = 15.98cm^2$

If we were only required to find the area of the triangle rather than all the included angles we could have used the alternative formula for the area directly after finding side c, as follows. Then, knowing the three sides we may use the formula

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where, } s = \frac{(8.2+4.5+7.11)}{2} = 9.905cm$$

$$A = \sqrt{9.905(9.905-8.2)(9.905-4.5)(9.905-7.11)}$$

So that the area $= 15.97cm^2$.

This is the same result as before, within the accuracy of the figures, used.

Example 1.3

Convert the point (3, 4) to polar coordinates *and* convert the point $\left(3,\ \pi/6\right)$ to Cartesian coordinates

Using formula 3.2 we have for point (3, 4),

$r = \sqrt{3^2+4^2} = 5$ and from $\theta = \tan^{-1}\left[\frac{y}{x}\right]$ we find that $\theta = \tan^{-1}\left[\frac{4}{3}\right] = \tan^{-1}1.333 = 53.1^\circ$

So point (3, 4) in polar form is $(5,\ 53.1^\circ)$.

Also, for the point $\left(3,\ \pi/6\right)$ where $\pi/6\ rad = 30^\circ$ we find from formula 3.1 that:

$x = r\cos\theta = 3\cos 30 = 2.6$ and

$y = r\sin\theta = 3\sin 30 = 1.5$. So the point $\left(3,\ \pi/6\right)$ in Cartesian form is (2.6, 1.5)

Example 1.4

Where the centre and what is the radius of the circle represented by the equation,

$x^2+y^2+4x-12y-9=0$

Then from equation 4.2, we find that:

$2g = 4$ and $2f = -12$, therefore $g = 2$ and $f = -6$ so that centre $(-g, -f) = (-1, 6)$. Also, $c = -9$

therefore the radius $= \sqrt{g^2+f^2-c}$ and so

$r = \sqrt{4+36+9} = \sqrt{49} = 7$

The application of more of these fundamental formulae will be shown in later examples, together with the use of some of the trigonometric identities that are listed next.

2. Trigonometric Identities

Listed below there is set out a fairly comprehensive list of trigonometric identities, many of which you will find useful when manipulating expressions that result from the analysis of engineering applications.

Formulae

1. General identities

1.1 for all values of θ: $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

1.2 $\sin^2\theta + \cos^2\theta = 1$

1.3 $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

1.4 $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$; $\cot\theta = \dfrac{1}{\tan\theta}$

1.5 $\tan^2\theta + 1 = \sec^2\theta$, $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

1.6 $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$

1.7 $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

1.8 $\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

2. Products to sums

2.1 $\sin A\cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$

2.2 $\cos A\sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$

2.3 $\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$

2.4 $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B)-\cos(A-B)\right]$

3. Sums/differences to products

3.1 $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

3.2 $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

3.3 $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

3.4 $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

Where $A > B$

4. Doubles and squares

4.1 From equation (1.6) $\sin 2A = 2\sin A\cos A$

4.2 From equations (1.2 and 1.7),

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

4.3 From equations (1.8), $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

4.4 $\sin^2 A + \cos^2 A = 1$ (see equation 1.2)

4.5 $\sec^2 A = 1 + \tan^2 A$ (see equation 1.5)

4.6 $\operatorname{cosec}^2 A = 1 + \cot^2 A$ (see equation 1.5)

5. Hyperbolic doubles and squares

Compare with equations (4.1 to 4.6) where:

$\coth A = \frac{1}{\tanh A}$, $\operatorname{sech} A = \frac{1}{\cosh A}$,

$\operatorname{cosech} A = \frac{1}{\sinh A}$

5.1 $\sinh A = 2\sinh A\cosh A$

5.2 $\cos 2A = \cosh^2 A - \sinh^2 A$

$$= 2\cosh^2 A - 1$$
$$= 1 + 2\sinh^2 A$$

5.3 $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

5.4 $\cosh^2 A - \sinh^2 A = 1$

5.5 $\operatorname{sech}^2 A = 1 - \tanh^2 A$

5.6 $\operatorname{cosech}^2 A = \coth^2 A - 1$

6. Halves and hyperbolic halves

When $t = \tan\frac{\theta}{2}$ then:

6.1 $\sin\theta = \frac{2t}{1+t^2}$

6.2 $\cos\theta = \frac{1-t^2}{1+t^2}$

6.3 $\tan\theta = \frac{2t}{1-t^2}$

When $t = \tanh\frac{\theta}{2}$ then:

6.4 $\sinh\theta = \frac{2t}{1-t^2}$

6.5 $\cosh\theta = \frac{1+t^2}{1-t^2}$

6.6 $\tanh\theta = \frac{2t}{1+t^2}$

The following examples show how trigonometric identities may be used to solve engineering problems and trigonometric equations.

Example 2.1

If the ratios of the sides of a triangle are such that $\sin\theta = \frac{3}{5}$. Find the ratios for $\cos\theta$ and $\tan\theta$, without references to tables or use of a calculator.

Then from formula (1.2), $\cos^2\theta = 1 - \sin^2\theta$ so that $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

We also know from formula 1.3 that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

therefore $\tan\theta = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

Example 2.2

In simple harmonic motion the variation of velocity of a point Q is given by $Q = -\omega r\sin\omega t$. With reference to Figure 11.2 on page 207 of the book, show that $Q = \pm\sqrt{r^2 - x^2}$.

You need to know that the angular displacement in radian is $\theta = \omega t$ and that the linear velocity v is equal to the product of the angular velocity ω and its radius r from the centre (Figure 11.2a), that is $v = \omega r$.

Then $Q = -\omega r \sin \omega t$ or $Q = -\omega r \sin \theta$ and from formula 1.2 where $\sin^2 \theta = 1 - \cos^2 \theta$ then,
$Q^2 = \omega^2 r^2 (1 - \cos^2 \theta)$ or $Q = \pm \omega r \sqrt{1 - \cos^2 \theta}$.

Now from Figure 11.2a), $\cos \theta = \dfrac{x}{r}$ so

$Q = \pm \omega r \sqrt{1 - \left(\dfrac{x}{r}\right)^2}$ and placing r inside the square root gives $Q = \pm \omega \sqrt{r^2 - x^2}$ as required.

In the following examples we verify trigonometric expressions and solve trigonometric equations using identities.

Example 2.3

Verify the following identities by showing that each side of the equation is equal in all respects:

1) $(\sin \theta + \cos \theta)^2 \equiv 1 + \sin 2\theta$

2) $\dfrac{\sin 3\theta - \sin \theta}{\cos \theta - \cos 3\theta} \equiv \cot 2\theta$

Where, the sign $\equiv$ means always equal to.

Then for identity (1) we are required to simply manipulate the LHS to equal the RHS or vice versa. So multiplying out the LHS of
$(\sin \theta + \cos \theta)^2 \equiv 1 + \sin 2\theta$ gives,
$\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \equiv 1 + \sin 2\theta$
$\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta \equiv 1 + \sin 2\theta$
Then, from formula 1.2, where, $\sin^2 \theta + \cos^2 \theta = 1$ we find that $1 + 2 \sin \theta \cos \theta \equiv 1 + \sin 2\theta$
Finally, from formula 4.1 where in this case $\sin 2\theta = 2 \sin \theta \cos \theta$ then,
$1 + \sin 2\theta \equiv 1 + \sin 2\theta$ as required.

2) Again considering LHS and using sums to products formulae 3.2 and 3.4 where,
$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$ (3.2)

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$ (3.4) where $A > B$ then

$$\sin 3\theta - \sin \theta = 2 \cos \left(\frac{3+1}{2}\right)\theta \sin \left(\frac{3-1}{2}\right)\theta$$
$$= 2 \cos 2\theta \sin \theta \text{ and}$$
$$\cos \theta - \cos 3\theta = -2 \sin \left(\frac{1-3}{2}\right)\theta \sin \left(\frac{1+3}{2}\right)\theta$$
$$= -2 \sin(-\theta)(\sin 2\theta)$$

And from formula (1.1) where, $\sin(-\theta) = -\sin \theta$ then $\cos \theta - \cos 3\theta = 2 \sin 2\theta \sin \theta$, therefore

$$\frac{\sin 3\theta - \sin \theta}{\cos \theta - \cos 3\theta} \equiv \frac{2 \cos 2\theta \sin \theta}{2 \sin 2\theta \sin \theta} \equiv \cot 2\theta$$

Example 2.4

If A is an acute angle and B is obtuse, where $\sin A = \dfrac{3}{5}$ and $\cos B = -\dfrac{5}{13}$, find the values of:
1) $\sin(A+B)$, 2) $\sin 2A$ and 3) $\tan(A+B)$.

1) In this case, in order to help find values for this relationship we may use the identity (1.6) i.e.:
$\sin(A+B) = \sin A \cos B + \cos A \sin B$ ....(1)
However to use identity (1) we need first, to find the ratios for $\sin A$ or $\cos B$, in terms of each other.

We know that $\sin^2 B = 1 - \cos^2 B$ (from formula 4.4) so:

$$\sin^2 B = 1 - \left(-\frac{5}{13}\right)^2$$
$$= 1 - \frac{25}{169} = \frac{144}{169}$$

Then, $\sin B = \dfrac{12}{13}$ (since B is obtuse $90^\circ < B < 180^\circ$ and sign ratio is positive in the second quadrant, then only positive values of the ratio need be considered).

Similarly from formula 4.4 $\cos^2 A = 1 - \sin^2 A$ so $\cos^2 A = 1 - \dfrac{9}{25}$ and $\cos A = \dfrac{4}{5}$ (since we are told that $\angle A$ is $< 90^\circ$ only the positive value need be considered).
Now using the identity we labelled (1) from above, then:

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A+B) = \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = -\frac{15}{65} + \frac{48}{65}$

so that $\sin(A+B) = \frac{33}{65}$.

Note the use of fractions to keep exact ratios.

2) Now to find the value for $(\sin 2A)$, we again need an identity that relates double and single angles formula 4.2 i.e., $\sin 2A = 2\sin A\cos A$ provides just such a relationship. Then using the values we found in part (1):

$$\sin 2A = 2\left(\frac{3}{5}\right)\left(\frac{5}{4}\right) = \frac{24}{25}.$$

3) For this part of the question we simply need to remember identity (1.3) i.e. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and use identity 1.8 in a similar way as before.

Then $\tan A = \frac{\sin A}{\cos A} = \frac{\frac{3}{5}}{\frac{4}{5}} = \left(\frac{3}{5}\right)\left(\frac{5}{4}\right) = \frac{3}{4}$ and

$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{12}{13}}{-\frac{5}{13}} = \left(\frac{12}{13}\right)\left(-\frac{13}{5}\right) = -\frac{12}{5}$$

Now, using identity 1.8 we obtain,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} = \frac{\frac{3}{4} - \frac{12}{5}}{1 - \left(\frac{3}{4}\right)\left(-\frac{12}{5}\right)}$$

Then multiplying every term in this expression by 20 (the lowest common multiple) we find that,

$$\tan(A+B) = \frac{15-48}{20+36} = -\frac{33}{56}.$$

Example 2.4

Solve the equation $2\sin\theta + \cos\theta = 2$ for angles θ between 0 and 2π (radian). Now, although not at all obvious, this equation may be solved using the half-angle identities, where using equations 6.1 $\sin\theta = \frac{2t}{1+t^2}$ and 6.2, $\cos\theta = \frac{1-t^2}{1+t^2}$ where $t = \tan\frac{\theta}{2}$ the above equation may be found in terms of the single variable (t), as follows.

$2\sin\theta + \cos\theta = 2$

$\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} = 2$ and on multiplication by $1+t^2$ we obtain $4t + 1 - t^2 = 2 + 2t^2$ or $-3t^2 + 4t - 1 = 0$. This is a quadratic that can be solved by factorisation.

Where $(-3t+1)(t-1) = 0$, so that $t = \frac{1}{3}$ or $t = 1$ and

since $t = \tan\frac{\theta}{2}$ then $\tan\frac{\theta}{2} = \frac{1}{3}$ giving $\frac{\theta}{2} = 18.43^\circ$

and $\tan\frac{\theta}{2} = 1$ giving $\frac{\theta}{2} = 45^\circ$. So in our range, $(0 - 2\pi)$ or $(0 - 360^\circ)$, $\theta = 36.87^\circ$ or $\theta = 90^\circ$.

Note the next values of $\frac{\theta}{2}$ corresponding to $\frac{1}{3}$ and 1 are, 198.43° and 225° which when doubled give values of θ that are outside of the required range.

We conclude our short study of trigonometric functions, with one more example concerning hyperbolic identities, that includes the use of the hyperbolic functions formulae (13 to 17) given previously in EM – Algebra.

Example 2.5

Show that the following hyperbolic identities are valid:

1) $\cosh^2 A - \sinh^2 A = 1$
2) $\mathrm{sech}^2 A = 1 - \tanh^2 A$
3) $\mathrm{cosech}^2 A = \coth^2 A - 1$

1) From formulae (13 – 15) in EM (Algebra) we know that:

$$\cosh A + \sinh A = \left(\frac{e^A + e^{-A}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right) = e^A$$

$$\cosh A - \sinh A = \left(\frac{e^A + e^{-A}}{2}\right) - \left(\frac{e^A - e^{-A}}{2}\right) = e^{-A}$$

Then remembering the factors for the difference of two squares, we may write that:

$$(\cosh A + \sinh A)(\cosh A - \sinh A) = (e^{A})(e^{-A})$$
$$= e^{0} = 1$$

$or\ \cosh^2 A - \sinh^2 A = 1 ....(i)$

2) If we divide each term in equation (i) by $\cosh^2 A$ we get:

$$\frac{\cosh^2 A}{\cosh^2 A} - \frac{\sinh^2 A}{\cosh^2 A} = \frac{1}{\cosh^2 A}$$

Then noting that $\frac{\sinh^2 A}{\cosh^2 A} = \tanh A$ (in a similar manner to equation 1.3 for tan A) and that from equations 5 where $\frac{1}{\cosh^2 A} = \mathrm{sech}^2 A$, we see that: $1 - \tanh^2 A = \mathrm{sech}^2 A$. Or $\mathrm{sech}^2 A = 1 - \tanh^2 A$, as required.

3) Similarly to that for part (2) if we now divide equation (i) by $\sinh^2 A$ we get:

$\frac{\cosh^2 A}{\sinh^2 A} - \frac{\sinh^2 A}{\sinh^2 A} = \frac{1}{\sinh^2 A}$. Now applying the same logic as in part (2) and remembering that $\mathrm{cosech}\, A = \frac{1}{\sinh A}$, then from the above we obtain the relationship: $\coth^2 A - 1 = \cos\mathrm{ech}^2 A$ or $\cos\mathrm{ech}^2 A = \coth^2 A - 1$ as required.